RUSSIAN ACADEMY OF SCIENCES

INSTITUTE FOR PROBLEMS IN MECHANICAL ENGINEERING

Proceedings of the XXX Summer School

ADVANCED PROBLEMS IN MECHANICS

APM '2002

St. Petersburg (Repino)
June 27 – July 6, 2002

Editor: D.A. Indeitsev

RAS

GAMM

St. Petersburg, 2003

On the steady motions of a sphere with a rotor on a plane with friction

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Abstract

The problem of the steady motions of a heavy non-homogeneous dynamically symmetric sphere with a rotor on a plane with sliding friction is considered. It is supposed, that the axis of a rotor coincides with an axis of dynamical symmetry of a sphere. Under this assumption the stability of regular precessions of the system is investigated.

Stability and bifurcations of steady motions of a non-homogeneous dynamically symmetric sphere (without rotor), moving along fixed horizontal plane with friction, are completely investigated in works [1, 2]. The basic results of these works contain in the monography [3]. In work [4] the stability of the steady motions of a sphere with fast-rotational rotor is investigated. Thus, in the present work the research begun in [1, 2, 4] is continued.

1 Formulation of a problem

Let non-homogeneous dynamically symmetric sphere move along fixed horizontal plane under the action of gravity. Let's assume, that a plane is rough, i.e. the reaction of a plane is the sum of the normal reaction and the force of sliding friction. Let's assume also, that inside a sphere there is a rotor, which rotates with a constant angular velocity about an axis of dynamical symmetry of a sphere. Under these assumptions the equations of motion of a sphere admite (see, for example, [5]) the non-increasing function – total mechanical energy of the system

$$2H = m\mathbf{v}^2 + A_1\left(\omega_1^2+\omega_2^2\right) + A_3\omega_3^2 - 2mga\gamma_3 \tag{1.1}$$

and two first integrals – generalized Jellett integral

$$K = A_1\left(\omega_1\gamma_1+\omega_2\gamma_2\right) + A_3\left(\omega_3+\sigma\right)\left(\gamma_3-\varepsilon\right) = k \tag{1.2}$$

and geometric integral

$$\Gamma = \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1. \tag{1.3}$$

Here m is the mass of a sphere; A_1 and A_3 are the principal central moments of inertia of a sphere; g is the acceleration due to gravity; σ is gyrostatic moment of a rotor referred to the axial moment of inertia of a sphere; $\varepsilon = a/r$ ($\varepsilon \in (0,1)$), where a is a distance from the geometrical centre of a sphere to its centre of mass along the axis of dynamical symmetry, which positive direction is chosen such, that $a > 0$, r is a radius of a sphere, v_i, ω_i, γ_i, ($i = 1, 2, 3$) are projections of vectors of velocity $\mathbf{v}$ of the centre of mass of a sphere, the angular velocity $\boldsymbol{\omega}$ and the unit vector $\boldsymbol{\gamma}$ of ascending vertical onto the principal central axes of inertia of a sphere. We denote also the unit vector of an axis of dynamical symmetry of a sphere by $\mathbf{e}$.

Geometric integral (1.3) can be considered as configuration space of essential coordinates $\boldsymbol{\gamma} \in \mathbf{S}^2$, where $\mathbf{S}^2$ is two-dimensional sphere named as Poisson sphere. Thus, the equations of motion of a sphere with a rotor admite non-increasing function (1.1) and linear (with respect to quasi-velocities $\boldsymbol{\omega}$) first integral (1.2). Hence, steady motions of this mechanical system can be investigated through modified Routh – Salvadori theory [3]. Taking into account that fact, that integral (1.2) is linear in quasi-velocities, it can be shown (see [3, 6]), that investigation of steady motions of a sphere with a rotor is reduced to study of the effective potential of the system. The effective potential is a minimum of function (1.1) with respect to generalized velocities v_i and ω_i ($i = 1, 2, 3$) at fixed level of the first integral (1.2). Its critical points on Poisson sphere $\mathbf{S}^2$ determined by geometric integral (1.3) correspond to steady motions of the system, and its point of minimum correspond to stable steady motions.

2 Construction of the effective potential and its analysis

Let $W(\boldsymbol{\gamma})$ be a minimum of the function $H(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\gamma})$ (function (1.1)) with respect to variables $\mathbf{v}$ and $\boldsymbol{\omega}$ at fixed level of Jellett integral (1.2):

$$W(\boldsymbol{\gamma}) = \min_{\mathbf{v},\boldsymbol{\omega}} H\big|_{K=k} = H\left(\mathbf{v}_k, \boldsymbol{\omega}_k, \boldsymbol{\gamma}\right),$$

$$\mathbf{v}_k = 0, \boldsymbol{\omega}_k = \lambda\left(\boldsymbol{\gamma} - \varepsilon\mathbf{e}\right),$$

$$\lambda = \left[k - A_3\sigma\left(\gamma_3-\varepsilon\right)\right]\left[A_1\left(1-\gamma_3^2\right) + A_3\left(\gamma_3-\varepsilon\right)^2\right]^{-1}.$$

In the explicit form we can write the effective potential as follows:

$$W(\gamma_3) = -mga\gamma_3 + \frac{1}{2}\frac{\left[k - A_3\sigma(\gamma_3 - \varepsilon)\right]^2}{A_1(1-\gamma_3^2) + A_3(\gamma_3 - \varepsilon)^2} = mgaf(\gamma_3),$$

$$f(\gamma_3) = -\gamma_3 + \frac{1}{2}\frac{\left[p - s(\gamma_3 - \varepsilon)\right]^2}{\delta(1-\gamma_3^2) + (\gamma_3 - \varepsilon)^2},$$

$$\left(s = \sigma\sqrt{\frac{A_3}{mga}},\ \delta = \frac{A_1}{A_3} \in (\frac{1}{2} + \infty),\ p = \frac{k}{\sqrt{mgaA_3}}\right).$$

It is obvious, that the function $W(\gamma_3)$ have critical values on sphere $\mathbf{S}^2$ in points $P_\pm = (\gamma_1 = \gamma_2 = 0, \gamma_3 = \pm 1)$ – poles of Poisson sphere, which correspond to permanent rotations of a sphere about its vertically situated axis of dynamical symmetry:

$$Q_\pm = (\omega_1 = \omega_2 = 0, \omega_3 = \omega_\pm; \gamma_1 = \gamma_2 = 0, \gamma_3 = \pm 1; \mathrm{v} = 0), \tag{2.1}$$

where $\omega_\pm$ is determined from a condition

$$k = A_3(\omega_\pm + \sigma)(\pm 1 - \varepsilon).$$

The stability of steady motions (2.1) was investigated in works [4, 5].

Besides the solutions (2.1) there are also other steady motions determined by relations

$$Q_\theta = \left(\omega_1 = \omega_\theta\gamma_1, \omega_2 = \omega_\theta\gamma_2, \omega_3 = \omega_\theta(\cos\theta - \varepsilon); \gamma_1^2 + \gamma_2^2 = \sin^2\theta, \gamma_3 = \cos\theta; \mathrm{v} = 0\right) \tag{2.2}$$

where ω_θ is determined from a condition

$$k = \left[A_1\sin^2\theta + A_3(\cos\theta - \varepsilon)^2\right]\omega_\theta + A_3\sigma(\cos\theta - \varepsilon),$$

and an angle θ is determined from the equation $df/d\theta = 0$.

The solutions of the equation $df/d\theta = 0$ correspond to parallels $P_\theta = \left(\gamma_1^2 + \gamma_2^2 = \sin^2\theta,\ \gamma_3 = \cos\theta\right)$ of Poisson sphere and solutions Q_θ correspond to regular precessions of a sphere: the sphere rotates with constant angular velocity $-\omega_\theta\varepsilon$ about its axis of dynamical symmetry, which rotates with constant angular velocity ω_θ about the vertical and the angle between an axis of dynamical symmetry and vertical is constantly equal θ.

The equation $df/d\theta = 0$ in the explicit form may be written as follows

$$Gp^2 - (E + FG)ps + EFs^2 - (E - FG)^2 = 0, \tag{2.3}$$

$$E = \delta(1 - \varepsilon\cos\theta),\ F = \cos\theta - \varepsilon,\ G = \varepsilon + (\delta - 1)\cos\theta,$$
$$E - FG = \delta\sin^2\theta + (\cos\theta - \varepsilon)^2 > 0.$$

The equation (2.3) is squared both with respect to p, and with respect to s. For its solvability with respect to p, the inequality

$$s^2 + 4G \geq 0. \tag{2.4}$$

should be satisfied.

Condition of stability of steady motions (2.2), obtained on the basis of the modified Routh – Salvadori theorem [3], has a form

$$(\delta - 1)(p - Fs)^2 + (E - FG)\left(s^2 + 4G\right) \geq 0. \tag{2.5}$$

Taking into account a condition (2.4) and the fact, that $E - FG > 0$ (see (2.3)), it is easy to see, that for $\delta > 1$ all the regular precessions of a sphere with a rotor will be stable. Thus, it is necessary to investigate the stability of regular precessions of a sphere for $\delta < 1$. Therefore, further we shall consider the case, when $1 - \delta > 0$.

3 The basic conclusions on stability for the case $\delta < 1$

First of all, let's solve the equation (2.3) with respect to p

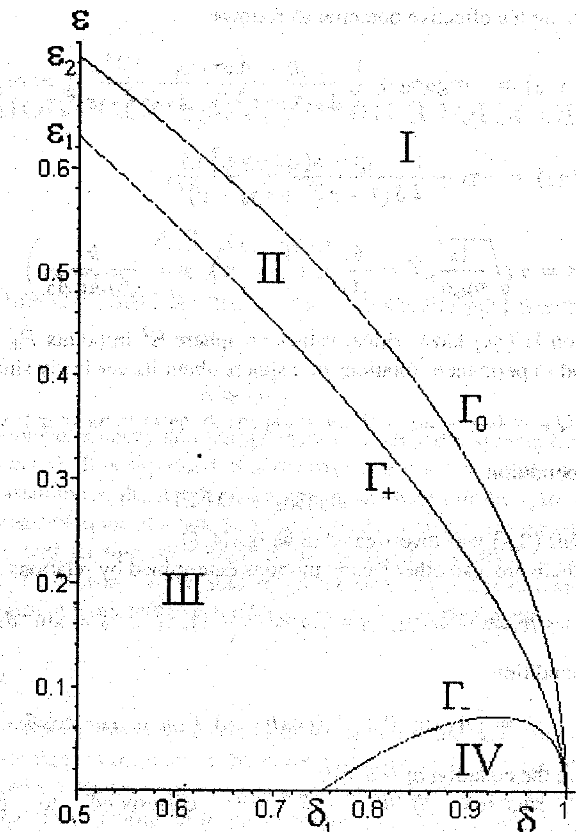
$$p_\pm = \frac{(E + FG)s \pm (E - FG)\sqrt{s^2 + 4G}}{2G}$$

and substitute these expressions to the condition of stability (2.5). After substitution we obtain the following inequality

$$\left(G^2 - L\right)s^2 + 2G\left(3G^2 - L\right) \geq \pm\left(G^2 + L\right)s\sqrt{s^2 + 4G}, \tag{3.1}$$

$$\left(G^2 + L\right) = (1 - \delta)(E - FG),\ L = \delta\left(1 - \delta - \varepsilon^2\right).$$

Further we shall distinguish cases, when $p = p_+$ and $p = p_-$. We shall denote the corresponding families of regular precessions by Q_θ^+ and Q_θ^-.



Let's consider a plane of parameters of a sphere (δ, ε) and let's allocate on this plane the following areas: the area I (is limited to beams $\delta = 1$ and $\varepsilon > \varepsilon_2$, $\delta = \delta_0$ and curve Γ_0); the area II (is limited to an intercept of a straight line $\varepsilon_1 < \varepsilon < \varepsilon_2$, $\delta = \delta_0$ and curves Γ_0 and Γ_+); the area III (is limited to an intercepts of straight lines $\varepsilon = 0$, $\delta_0 < \delta < \delta_1$ and $0 < \varepsilon < \varepsilon_1$, $\delta = \delta_0$ and curves Γ_+ and Γ_-); the area IV (is limited to an intercept of straight line $\varepsilon = 0$, $\delta_1 < \delta < 1$ and curve Γ_-) (see figure). Constant values δ_0, δ_1, ε_1 and ε_2 are equal, respectively

$$\delta_0 = \frac{1}{2}, \ \delta_1 = \frac{3}{4}, \ \varepsilon_1 = \frac{3+\sqrt{2}}{7}, \ \varepsilon_2 = \frac{1}{\sqrt{2}},$$

and the curves Γ_0, Γ_+ and Γ_- are determined by the equations

$$\begin{array}{lcl} \Gamma_0 & : & \delta = 1 - \varepsilon^2, \\ \Gamma_+ & : & 4\delta^2 - (1-\varepsilon)(7+\varepsilon)\,\delta + 3\,(1-\varepsilon)^2 = 0, \\ \Gamma_- & : & 4\delta^2 - (1+\varepsilon)(7-\varepsilon)\,\delta + 3\,(1+\varepsilon)^2 = 0. \end{array}$$

Detailed analysis of an inequality (3.1) allows to make following conclusions on stability of steady motions of a sphere:

1. In area I for all values s all the regular precessions of a sphere with a rotor (both the family Q_θ^+, and family Q_θ^-) will be stable;
2. In area II precessions Q_θ^+ are steady for $s^2 < s_*^2$ and unstable for $s^2 > s_*^2$, and precessions Q_θ^- are stable for all values of s;
3. In area III
   (a) If $\cos\theta > c_*$, then precessions Q_θ^+ are unstable for all values of s, and precessions Q_θ^- are stable for $s^2 > s_*^2$ and unstable for $s^2 < s_*^2$;
   (b) If $\cos\theta < c_*$, then precessions Q_θ^+ are stable for $s^2 < s_*^2$ and unstable for $s^2 > s_*^2$, and precessions Q_θ^- are stable for all values of s;
4. In area IV precessions Q_θ^+ are unstable for all values of s, and precessions Q_θ^- are stable for $s^2 > s_*^2$ and unstable for $s^2 < s_*^2$.

Here we denote by s_*^2 and c_* the following expressions:

$$s_*^2 = s_*^2(\delta, \varepsilon, \theta) = \frac{G\left(G^2 - 3L\right) + \left(G^2 + L\right)\sqrt{G^2 + L}}{L},$$

$$c_* = c_*(\delta, \varepsilon) = \frac{\varepsilon}{1 - \delta} - \frac{\sqrt{L}}{\sqrt{3}\,(1 - \delta)}.$$

Thus, stability of regular precessions of a sphere with a rotor on a plane with friction is completely investigated. The obtained results will well be coordinated with known results of works [1, 2, 4].

Acknowledgements

The author is grateful to Prof. A.V. Karapetyan and Prof. V.E.Pavlovsky for support of work and useful discussions.

This research was supported by the Russian Foundations for Basic Research (00-15-96150, 02-01-06041, 02-01-00750).

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