Analytical description of low-Tc DC SQUID response and methods for its linearization

I.I. Soloviev*, N.V. Klenov, A.E. Schegolev, S.V. Bakurskiy, and M.Yu. Kupriyanov
Lomonosov Moscow State University, Skobeltsyn Institute of Nuclear Physics
Moscow, Russia
Email: isol@phys.msu.ru

M.V. Tereshonok
Moscow Technical University of Communications and Informatics
Moscow, Russia

A.A. Golubov
Moscow Institute of Physics and Technology (State University)
Dolgoprudny, Russia

***Abstract*—We derived inductance-dependent expressions for time-averaged voltage-flux and voltage-current dependencies of Low-Tc DC SQUID. Utilization of these expressions speeds up optimization of real devices and expands possibilities for their analytical study. Optimization of voltage response linearity of two SQUIDs connected in series is presented as an example of application of the proposed analytics.**

Keywords—DC SQUID; voltage response; linearity

I. INTRODUCTION

Progress of modern SQUID fabrication technology based on low-temperature superconductors allows precise control of its parameters. This expands the area of SQUID application but also tightens requirements to the circuit. For example, linearity of DC SQUID response becomes important in applications where SQUID should ideally act as a linear broadband magnetic flux-to-voltage transformer, like susceptometry or nuclear magnetic resonance measurements, electrically small antennas or frequency division based read-out of matrix of nanoscale devices such as photon detectors or NEMS.

In this study we present analytical description of voltage response of DC SQUID with practical parameters. Since the mentioned applications imply the using of overdamped Josephson junctions, we neglect the junction's capacity in our model. The noise effect is omitted for simplicity. Therefore our results are valid if the Josephson energy E_J is much higher than the energy E_f of thermal fluctuations, $E_J / E_f \leq 10^{-3}$. The obtained expressions describing DC SQUID flux-to-voltage transformation can be directly used in various methods suggested for voltage response linearization. According example is presented in the last part of this paper.

I.I.S., N.V.K., A.E.S., S.V.B., M.Yu.K were supported by grant No. 17-12-01079 of Russian Science Foundation, M.V.T. was supported by RFBR grant No. 16-29-09515-ofi-m and grant for leading scientific school No. 8168.2016.2, A.A.G. is supported by grant No. 14.Y26.31.0007 of Ministry of Education and Science of the Russian Federation.

II. SQUID VOLTAGE

A. Analytical Expressions for SQUID with Small Inductance

In the frame of the resistively shunted junction model the system of equations describing DC SQUID has the form:

$$d\varphi_+ / d\tau = i_b - \sin(\varphi_+) \cos(\varphi_-), \quad (1)$$

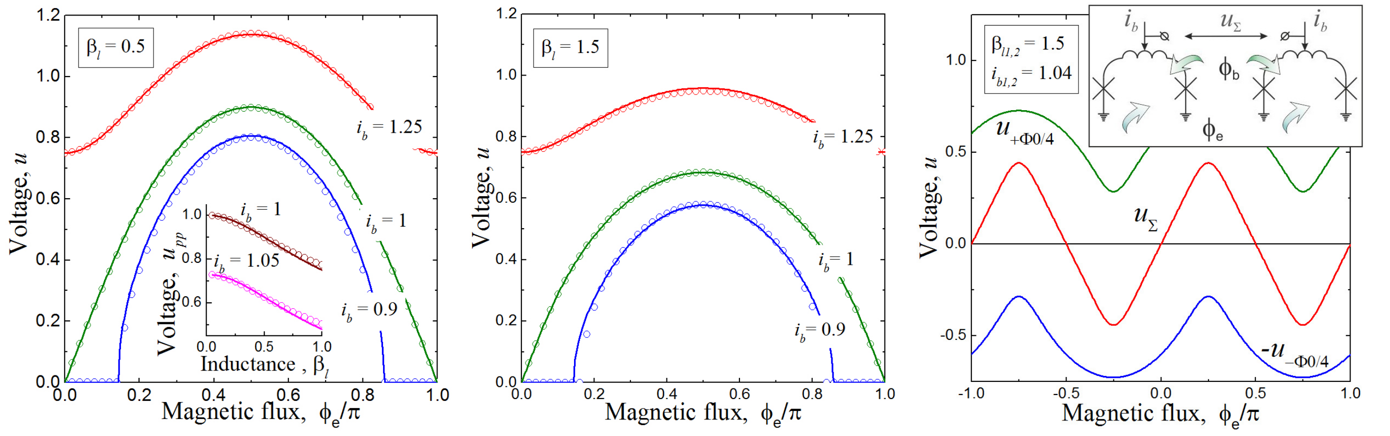
$$d\varphi_- / d\tau = -(\varphi_- - \phi_e) / \beta_l - \sin(\varphi_-) \cos(\varphi_+), \quad (2)$$

where $\varphi_\pm = (\phi_1 \pm \phi_2) / 2$, $\phi_{1,2}$ are the Josephson junctions phases, time $\tau = t\omega_c$ ($\omega_c = 2\pi I_c R_n / \Phi_0$ is the characteristic frequency, I_c is the Josephson junction critical current, R_n is the shunt resistance and $\Phi_0 = h / 2e$ is the flux quantum, h is the Planck constant and e is the electron charge), $i_b = I_b / 2I_c$ is the normalized bias current, $\beta_l = \pi L I_c / \Phi_0$ is the normalized SQUID inductance, and $\phi_e = \pi\Phi_e / \Phi_0$ is the normalized applied magnetic flux.

The well-known description of DC SQUID voltage response [1] was elaborated for the circuit with zero inductance, $\beta_l = 0$. The Josephson junctions phase difference φ_- is equal to the applied magnetic flux ϕ_e in this case (see (2)). This allows one to find the phase sum φ_+ and its time derivative from (1). Time averaged voltage, $u_0 = \langle d\varphi_+ / d\tau \rangle$, here is given by expression:

$$u_0 = [i_b^2 - \cos^2(\phi_e)]^{0.5}. \quad (3)$$

With bringing inductance into consideration we introduce time dependence of the phase difference φ_-. This dependence arises because of finite time required for the bias current redistribution between non-simultaneously switching Josephson junctions at non-zero applied magnetic flux. Redistributed part of the bias current circulating in the SQUID



a) b) c)

Fig. 1 DC SQUID flux-to-voltage transfer function (a), (b) and voltage response peak-to-peak amplitude (inset (a)) obtained by expressions (6), (7) (a) and (8) (b) (lines), and calculated numerically using (1), (2) (dots). Voltage responses of SQUIDs biased by magnetic flux $\pm\Phi_0/4$ in differential scheme and their total response (c); inset shows differential connection of SQUIDs.

loop, i_{cir}, is equal to ratio of oscillating part of the phase difference to the SQUID inductance, $i_{cir} = \varphi_l / \beta_l$.

If the SQUID inductance is small, $\beta_l \ll 1$, then the oscillating part of the phase difference is assumed to be small too, $\varphi_l \ll 1$. The phase difference can be presented as a sum of constant and small oscillating terms, $\varphi_- = \phi_e + \varphi_l$, accordingly. This allows to transform (2) into equation for φ_l determination:

$$d\varphi_l / d\tau = -\varphi_l / \beta_l - [\sin(\phi_e) + \varphi_l \cos(\phi_e)] \cos(\varphi_+). \quad (4)$$

One can use φ_+ found previously (for $\beta_l = 0$) to find φ_l from (4) assuming that the phase sum is not affected significantly by introduction of small inductance. Time averaging of according term in (1) gives correction to the voltage arising due to finite inductance:

$$u_l = \sin(\phi_e) < \varphi_l \sin(\varphi_+) >. \quad (5)$$

Explicit expression for the total voltage, $u = u_0 + u_l$, is then as follows:

$$u = u_0 - [1 + (\beta_l u_0)^{-2}]^{-1} (i_b - u_0) \tan^2(\phi_e). \quad (6)$$

Peak-to-peak amplitude of the voltage response, $u_{pp} = u[\phi_e = \pi] - u[\phi_e = 0]$, can be found from (6) directly:

$$u_{pp} = i_b (1 - 0.5 [i_b^2 + \beta_l^{-2}]^{-1}) - (i_b^2 - 1)^{0.5}. \quad (7)$$

Expressions (6), (7) describe inductance-dependent voltage-flux and voltage-current characteristics which can be utilized, e.g., in optimization of nanoSQUIDs.

Fig. 1a shows the voltage response u and its peak-to-peak amplitude u_{pp} dependence on inductance in the main panel and in the inset, correspondingly. The data are obtained making use of (6), (7) (lines) and by numerical calculations of (1), (2) (dots). It is seen that for the chosen inductance, $\beta_l = 0.5$, the curves are well consistent for the bias current around the critical current.

Analytical curves for the voltage response peak-to-peak amplitude deviate from the numerical ones starting from $\beta_l \approx 1$. This shows that linearization of (1), (2) performed under assumption $\varphi_l \ll 1$ is not valid for higher inductance values.

B. Expressions for SQUID with Moderate Inductance

While analytical solution of the system (1), (2) with moderate values of inductance, $\beta_l = 1 \ldots 3$, can hardly be derived, one can use (6) as a fitting function,

$$u = u_0 - a [1 + (\beta_l^* u_0)^{-2}]^{-1} (i_b - u_0) \tan^2(\phi_e), \quad (8)$$

for numerically calculated voltage response. Here we introduce a couple of fitting parameters, which are the amplitude, a, of the inductance-dependent part of the voltage, and the effective inductance, β_l^*, which is substituted into (6) instead of the real one.

Analytical expressions for the fitting parameters are rather complicated:

$$a = 2pq (2i_b^2 - 1) / \xi \quad \text{and} \quad \beta_l^* = (\xi / \chi)^{0.5}, \quad (9)$$

where

$$\xi = -0.586p + 2q + 4pq (i_b^2 - 1),$$

$$\chi = 0.586\, i_b^2 p - q - 2pq (i_b^2 - 1),$$

$$p = \beta_l^{1.66} / (2.154\, \beta_l^{1.48} + 2.285),$$

$$q = \beta_l^{1.92} / (4.28\, \beta_l^{1.625} + 5.06).$$

Details can be found in [2].

Solution (8) can be utilized in the most cases of SQUID or SQUID array applications mentioned in the introduction. Validity of (8) is limited by the range of inductance β_l where effective inductance β_l^* (9) is real. This range increases with increase of the bias current, and for $i_b = 1$ it is $\beta_l \in [0.2, 3.4]$.

Fig. 1b presents comparison of the data obtained by numerical calculations of (1), (2) (dots) and by using (8) (lines). One can see that the numerical data is fitted fairly well around the bias current equal to the critical current.

III. Linearization of SQUID Voltage Response

Let us begin our consideration of the SQUID voltage response linearity from the two limiting cases: (i) $i_b = 1$ and (ii) $i_b >> 1$, with $\beta_l = 0$ for simplicity. Here the voltage response is described correspondingly by the functions (see (3)):

$$\text{(i)}\ u_0 = |\sin(\phi_e)| \quad \text{and} \quad \text{(ii)}\ u_0 \approx B_+ - B_-\cos(2\phi_e), \quad (10)$$

where $B_\pm = (i_b \pm [i_b^2 - 1]^{0.5}) / 2$.

The highest voltage response amplitude can be obtained in both cases with additional bias flux, $\phi_b = \pi/4$ ($\Phi_b = \Phi_0/4$). However, in the first case this bias flux does not provide linearity. The bias flux value should be shifted toward zero, $\phi_b \rightarrow 0$, to make the response more linear [3], which in turn limits input (and output) signal amplitude.

While in the second case $\phi_b = \pi/4$ is optimal for linearity, the peak-to-peak voltage response is initially small due to high bias current. Therefore in general, one has to tune both parameters i_b and ϕ_b to find acceptable trade-off between linearity of the voltage response and its magnitude.

This trade-off can be overcome by differential connection of two SQUIDs suggested in [4]. Schematic of such connection is presented in inset of Fig.1c. Total voltage response of the differential scheme corresponds to subtraction of voltage responses of the two SQUIDs forming the scheme, $u_\Sigma = u_1 - u_2$. Application of equal in modulus but opposite in sign bias flux to each SQUID, $\phi_{b1} = -\phi_{b2} = \alpha$, provides mutual shift of the SQUIDs responses leading to non-zero total response.

The total response u_Σ can be easily found in the first limiting case (i) considered above, $i_{b1,2} = 1$, $\beta_{l1,2} = 0$. Inside the flux range $\phi_e \in [-\alpha, \alpha]$ (where $\alpha \leq \pi/2$) the voltage responses of the SQUIDs are $u_{1/2} = \pm\sin(\phi_e \pm \alpha)$, and therefore the total voltage response is

$$u_\Sigma = 2\cos(\alpha)\sin(\phi_e). \quad (11)$$

Peak-to-peak amplitude of the total voltage response is $u_{\Sigma pp} = 2\sin(2\alpha)$. For $\alpha = \pi/4$ this amplitude, $u_{\Sigma pp} = 2$, is just the same as for series connection of two SQUIDs. At the same time, the most linear place of the total response is now just in its center (at $\phi_e = 0$), and so the whole working slope of the response is available for flux-to-voltage transformation. We conclude that 2/3 of this slope is linear assuming that $\sin(\phi_e) \approx \phi_e$ for $\phi_e \in [-\pi/6, \pi/6]$. This is in contrast to conventional series connection of SQUIDs (cases (i), (ii)) where linear part is just 1/3 of the slope. Generally in comparison with series connection, differential connection enlarges linear part of the working slope, increases the linear part magnitude in voltage but decreases flux-to-voltage transfer coefficient (see (11)).

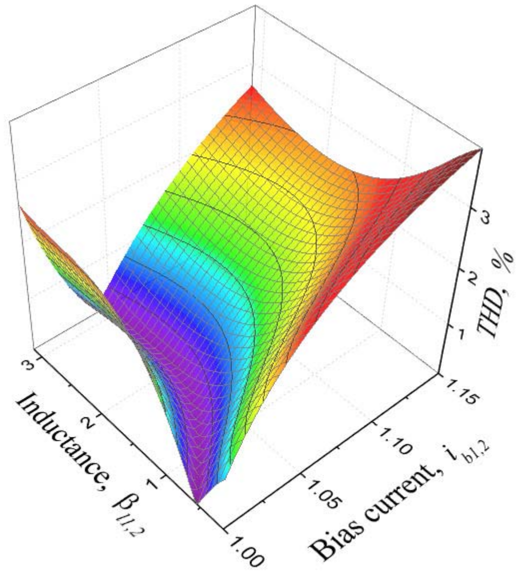


Fig. 2. Total harmonic distortion of harmonic signal applied to differential scheme of DC SQUIDs versus their bias current and inductance.

By using (8) we optimize the total voltage response of differential scheme in respect to total harmonic distortion (THD), varying $i_{b1,2}$ and $\beta_{l1,2}$ with fixed $\alpha = \pi/4$ and applied harmonic signal amplitude, $A = \pi/6$, see Fig. 2. The found optimal values of parameters, $i_{b1,2} = 1.04$ and $\beta_{l1,2} = 1.5$, provide THD = 0.2 %. Corresponding shapes of the total voltage response and the SQUIDs responses are shown in Fig.1c. The obtained THD value can be further improved by decrease of the bias flux. Another option is utilization of bi-SQUIDs [5] or parallel arrays of SQUIDs [6] with initially more linear response. They also can be used in differential scheme.

References

[1] A. Barone and G. Paterno, Physics and applications of the Josephson effect, New York: Wiley, 1982.

[2] I.I. Soloviev, N.V. Klenov, A.E. Schegolev, S.V. Bakurskiy, and M.Yu. Kupriyanov, "Analytical derivation of DC SQUID response," Supercond. Sci. Technol., vol. 29, p. 094005, 2016.

[3] M. Muck and J. Clarke, "Harmonic distortion and intermodulation products in the microstrip amplifier based on a superconducting quantum interference device," Appl. Phys. Lett., vol. 78, pp. 3666 - 3668, 2001.

[4] V. Kornev, I. Soloviev, N. Klenov, and O. Mukhanov, "High linearity josephson-junction array structures," Phys. C, vol. 468, pp. 813–816, 2008.

[5] V.K. Kornev, A.V. Sharafiev, I.I. Soloviev and O.A. Mukhanov, "Signal and noise characteristics of bi-SQUID", Supercond. Sci. Technol., vol. 27, p. 115009, 2014.

[6] V.K. Kornev, A.V. Sharafiev, I.I. Soloviev, N.V. Kolotinskiy, V.A. Scripka, O.A. Mukhanov, "Superconducting Quantum Arrays," IEEE Trans. Appl. Supercond., vol. 24, p. 1800606, August 2014.