The Limit Distribution of the Queue Length in a Priority System with Autoregressive Arrivals Under the Heavy Traffic Condition

Alexey Bergovin[1(✉)] and Vladimir Ushakov[1,2]

[1] Lomonosov Moscow State University, Moscow, Russia
alexey.bergovin@gmail.com

[2] The Institute of Informatics Problems of the Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences, Moscow, Russia

Abstract. In this paper a one-line queueing system with two priority classes, relative priority, Poissonian input flow with random intensity and infinite number of places in queue for waiting is considered. The current intensity value is taken at the beginning of the time reckoned for the arrival of the next requirement. Successive values of the flow intensity form a Markov chain of a special kind. This input flow structure allows to take in consideration not only mathematical expectation and variance, but also correlation between interval of two next arrivals. The main result is the limit distribution of the queue length for the least priority class, it is obtained in an explicit form. Also, analytical expressions for the density function, mathematical expectation and variance are given. Numerical examples, which show difference among limit distributions (for different parameters) for studied cases are provided.

Keywords: Poissonian flow · random intensity · relative priority · queue length · heavy traffic

1 Introduction

The main goal of this paper is to study the behaviour of the queue length of the lowest priority class in the queueing system with the autoregressive input flow (will be fully described in Sect. 2). The relevance of this research is determined by the fact that the vast majority of real service systems in many applied areas operate under conditions of either the heavy load or close to the heavy load. For instance, the widespread distribution of communication networks has led to a sharp increase in the volume of network traffic, and, therefore, an increase in the load on these networks, most of which have become highly loaded. It also lead us to the relevance of considering exactly this structure of the incoming flow. In [2,3] the authors have shown that there is a stochastic dependence between an inter-arrival time of two adjacent requests.

A. Dudin et al. (Eds.): ITMM 2023/WRQ 2023, CCIS 2163, pp. 19–32, 2024.
https://doi.org/10.1007/978-3-031-65385-8_2

The key purpose of this study is to find the limit distribution of the queue length of the lowest priority class. In addition, the probability density function, mathematical expectation and variance are given for the limit distribution. It should be mentioned that there is a huge amount of papers and monographs dedicated to studying queueing systems under the heavy traffic condition [4–16].

The paper has the following structure. In Sect. 2 the studied system is fully described. In Sect. 3, the results from [1], which will be used during the research, are given. Section 4 consists of two parts: in the first one some auxiliary expansions in a series are obtained, in the second one the main theorem is proved. In Sect. 5, some numerical examples are given.

2 System Definition

In this work sequence of queuing systems (the series scheme) is researched; m-th queueing system has the following structure. The structure of arrivals is time z_{m1} before the arrival of the first requirement and interval z_{mn} between $(n-1)$ - th and n-th requirement have an exponential distribution with random parameter $a_m^{(n)},\ n=1,2,\dots$. Value $a_m^{(n)}$ is selected just before the beginning of interval z_{mn} such that, $\mathbb{P}(a_m^{(1)}=a_{mj})=c_{mj},\ a_{mi}\neq a_{mj},\ i\neq j,\ c_{mj}>0,\ j=\overline{1,N},\ \sum\limits_{j=1}^{N} c_{mj}=1$ and $a_m^{(n)}=\xi_m\cdot a_m^{(n-1)}+(1-\xi_m)\cdot b_m^{(n)}$, where $b_m^{(n)}, n=1,2,\dots,$ $a_m^{(n)}, n=1,2,\dots$, and ξ_m are independent random variables. The distribution of the random variables $a_m^{(n)}$ and $b_m^{(n)}$ coincides with the distribution of $a_m^{(1)}$, $n=1,2,\dots$, and ξ_m has Bernoulli distribution with parameter p_m.

It is easy to show that

$$\mathbb{P}(z_{mn}<t)=\sum_{j=1}^{N} c_{mj}(1-e^{-a_{mj}t})$$

$$\mathbb{P}(z_{mn}<t_1, z_{m,n+k}<t_2)=(1-p_m^k)\sum_{j=1}^{N} c_{mj}(1-e^{-a_{mj}t_1})\sum_{k=1}^{N} c_{mk}(1-e^{-a_{mj}t_2})+ \\ +p_m^k\sum_{k=1}^{N} c_{mk}(1-e^{-a_{mj}t_1})(1-e^{-a_{mj}t_2})$$

$$\mathbb{E}z_{mn}=\sum_{j=1}^{N}\frac{c_{mj}}{a_{mj}},\quad \mathbb{D}z_{mn}=\sum_{j=1}^{N}\frac{c_{mj}}{a_{mj}^2},\quad corr(z_{mn},z_{m,n+k})=\frac{p_m^k}{2}\left(1-\frac{(\mathbb{E}z_{mn})^2}{\mathbb{D}z_{mn}}\right) \tag{1}$$

Further m will be used in indexes only where it is necessary to highlight dependence on m.

There are two special cases: $p=0$ and $p=1$, in the first one the input flow is hyper-exponential, in the second one: we obtain a system such that the initial intensity is randomly selected from the set $\{a_1,\dots,a_N\}$ with probabilities

$c_1, \ldots, c_N$ respectively, and afterward acts as a system with a Poissonian input flow with the chosen intensity (this case is not considered in this work).

From (1), it follows that if $\mu > 0$ and $\sigma > \mu$, there exists second-order flow ($N = 2$) such that it belongs to the class of flows considered in this work, and its expectation and variance of intervals between arrivals are equal to μ and σ^2, respectively. The correlation coefficient is equal to $\frac{p}{2}\left(1 - \left(\frac{\mu}{\sigma}\right)^2\right)$. While mathematical model of real system is being constructed, it is possible to adjust the first two moments of the real arrival process and their dependence.

All arriving requirements are divided into 2 classes with probabilities p_1, p_2 ($p_1 + p_2 = 1$), respectively, and it does not depend on other requirements. We firstly assume that each type of requirement forms its own queue. Secondly, if a service is started, it is never interrupted. The studied system operates under the relative priority discipline.

We assume that the system is free of requirements for $t = 0$ and serving lengths are independent random variables equally distributed for requirements of each particular type. The distribution function is $B_{mi}(x)$ and the density is $b_{mi}(x)$ for ith class and mth system , $i = 1, 2$; $\beta_{mi}(s)$ — Laplace-Stieltjes transform of function $b_{mi}(x), i = 1, 2$; β_{mij} – jth moment of random variable with $B_{mi}(x)$ distribution function.

$L(t) = (L_1(t), L_2(t))$ – amount of requirements in a system in time t.

It is known that if $\left(\sum\limits_{i=1}^{N} c_j a_j^{-1}\right)^{-1} \cdot (p_1\beta_{11} + p_2\beta_{21}) < 1$ the non-degenerate limit distribution of stochastic process $L(t)$ exists. In this work, $L_2(t)$ is studied in case while $t \to \infty$ and $\left(\sum\limits_{i=1}^{N} c_{mj} a_{mj}^{-1}\right)^{-1} \cdot (p_{m1}\beta_{m11} + p_{m2}\beta_{m21}) \to 1, m \to \infty$.

We will study the system under the next assumptions:

I) the first and the second moment of service time distribution exist (for each priority class) and

$$\beta_i(s) = 1 - \beta_{i1}s + \frac{\beta_{i2}}{2}s^2 + o_m(s^2), i = 1, 2,$$

where $o_m(s^2)/s^2 \to 0$ while $s \to 0$ uniformly on variable m

II) for each $m \in \{1, 2, \ldots\}$: $a_m(p_1\beta_{m11} + p_2\beta_{m21}) < 1$

III) following limits exist $\lim c_i = c_i^*$, $\lim a_i = a_i^*$, $\lim \beta_{ij} = \beta_{ij}^*$, $\lim p_j = p_j^*$, $i = \overline{1, N}$, $j = 1, 2$, where lim denote $\lim\limits_{m\to\infty}$.

The main goal of this study is to find

$$\lim_{m\to\infty} \mathbb{P}\left(\rho^\gamma \cdot L_2\left(\frac{t}{\rho^\alpha}\right) < x\right)$$

where

$$\rho = 1 - a(p_1\beta_{11} + p_2\beta_{21}), \quad a = \left(\sum_{i=1}^{N} \frac{c_j}{a_j}\right)^{-1}, \quad \gamma = \begin{cases} 0.5\alpha, & \alpha \leqslant 2, \\ 1, & \alpha > 2. \end{cases}$$

3 Preliminaries

In [1] expressions which Laplace-Stieltjes transform of generating function of queue length is satisfied has been found. They will be used to find the limit distribution. Let us write them.

Lemma 1. *Equation*

$$(1-p)z\sum_{m=1}^{N}\frac{a_m c_m}{\mu(z)+a_m\left(1-pz\right)}=1,$$

has N *continious in domain* $|z|\leqslant 1$ *solutions* $\mu=\mu_k(z),\ k=1,\ldots,\ N,$ *that*:

1) *only one function* $\mu_k(z)$ *is equal to 0 while* $z=1$;
2) $\Re(\mu_j(z))<0$ *for all* $j=1,\ldots,N$ *and* $|z|<1$;
3) $\mu_i(z)\neq\mu_j(z)$ *while* $i\neq j$.

Denote $\alpha_k(z)=\prod\limits_{j\neq k}[\mu_k(z)-\mu_j(z)]$.

Lemma 2. *For each* $k=1,\ldots,\ N$ *system of equations*

$$z_1=\beta_1(s-\mu_k(p_1z_1+p_2z_2)),$$

$$z_2=\beta_2(s-\mu_k(p_1z_1+p_2z_2)),$$

has a unique solution $z_i = z_{ik}(s)$, *such, that* $|z_{ik}(s)|<1$ *while* $k=2,\ldots,\ N,\ \Re s\geq 0,$ *and* $z_{i1}(0)=1,\ |z_{i1}(s)|<1$ *while* $\Re s>0,\ i=1,2.$

Lemma 3. *Laplace-Stieltjes transform of joint generating function of queue length for the first and the second classes is*

$$\begin{aligned}
p(z_1,z_2,s)&=p_0(s)+\\
&+\frac{p_1z_1+p_2z_2-1}{(1-p)(p_1z_1+p_2z_2)}\times\sum_{k=1}^{N}\frac{1}{\mu_k(p_1z_1+p_2z_2)(s-\mu_k(p_1z_1+p_2z_2))}\times\\
&\times\Big[\gamma_1^{(k)}(z_1,z_2,s)[1-\beta_1(s-\mu_k(p_1z_1+p_2z_2))]+\\
&+\gamma_2^{(k)}(z_1,z_2,s)[1-\beta_2(s-\mu_k(p_1z_1+p_2z_2))]\Big],
\end{aligned}$$

where functions $\gamma_i^{(k)}(z_1,z_2,s),\ i=1,2,\ k=\overline{1,N}$ *satisfy*:

$$\gamma_1^{(k)}(z_1,z_2,s)\frac{z_1-\beta_1(s-\mu_k(p_1z_1+p_2z_2))}{z_1}+ + \gamma_2^{(k)}(z_1,z_2,s)\frac{z_2-\beta_2(s-\mu_k(p_1z_1+p_2z_2))}{z_2}= = \frac{(1-p)(p_1z_1+p_2z_2)}{\alpha_k(p_1z_1+p_2z_2)}\prod_{m=1}^{N}[\mu_k(p_1z_1+p_2z_2)+a_m(1-p(p_1z_1+p_2z_2))]\times \times\sum_{j=1}^{N}\frac{c_ja_jf_j(z_1,z_2,s)}{\mu_k(p_1z_1+p_2z_2)+a_j(1-p(p_1z_1+p_2z_2))};$$

$$f_j(z_1,z_2,s)=1-(s+a_j(1-p(p_1z_1+p_2z_2))c_j^{-1}p_{0j}(s)+ + (p_1z_1+p_2z_2)(1-p)\sum_{k=1}^{N}a_kp_{0k}(s),\ j=\overline{1,N},$$

$$\gamma_1^{(k)}(z_1,z_2,s)=\frac{(1-p)(p_1z_1+p_2z_2)}{\alpha_k(p_1z_1+p_2z_2)}\prod_{j=1}^{N}[\mu_k(p_1z_1+p_2z_2)+a_j(1-p(p_1z_1+p_2z_2))]\times \times\sum_{e=1}^{N}\frac{a_ep_{1e}(z_1,z_2,0,s)}{\mu_k(p_1z_1+p_2z_2)+a_e(1-p(p_1z_1+p_2z_2))};$$

$$\gamma_2^{(k)}(z_1,z_2,s)=\frac{(1-p)(p_1z_1+p_2z_2)}{\alpha_k(p_1z_1+p_2z_2)}\prod_{j=1}^{N}[\mu_k(p_1z_1+p_2z_2)+a_j(1-p(p_1z_1+p_2z_2))]\times \times\sum_{e=1}^{N}\frac{a_ep_{2e}(z_2,0,s)}{\mu_k(p_1z_1+p_2z_2)+a_e(1-p(p_1z_1+p_2z_2))}.$$

functions $p_{0j}(s)$ *might be found from*:

$$p_{0j}(s)=\frac{1}{a_j}\sum_{l=1}^{N}\frac{1-p(p_1z_{l1}^*+p_2z_{l2}^*)}{(1-p)(p_1z_{l1}^*+p_2z_{l2}^*)(s-\mu_l^*(s))}\cdot\frac{1}{\prod_{n\neq j}(a_j-a_n)}\times \times\left(\frac{\mu_l^*(s)}{1-p(p_1z_{l1}^*+p_2z_{l2}^*)}+a_j\right)^{-1}\times \times\prod_{l\neq n}\left(\frac{\mu_l^*(s)}{1-p(p_1z_{l1}^*+p_2z_{l2}^*)}-\frac{\mu_n^*(s)}{1-p(p_1z_{n1}^*+p_2z_{n2}^*)}\right)^{-1}\times \times\prod_{k=1}^{N}\frac{(\mu_k^*(s)+a_j(1-p(p_1z_{k1}^*+p_2z_{k2}^*))(\mu_l^*(s)+a_k(1-p(p_1z_{l1}^*+p_2z_{l2}^*))}{(1-p(p_1z_{k1}^*+p_2z_{k2}^*))(1-p(p_1z_{l1}^*+p_2z_{l2}^*))},\quad (2)$$

where $\mu_k^*(s)=\mu_k(p_1z_{k1}^*+p_2z_{k2}^*)$

4 Main Result

To prove the main theorem, some auxiliary expansions in a series are needed, which would be formulated as separate lemmas.

4.1 Auxiliary Expansions in a Series

Lemma 4. *The next asymptotics for* $z(s\rho^\alpha)$ *are true*:

$$z(s\rho^\alpha)-1=\begin{cases}-\sqrt{\dfrac{s\rho^\alpha}{a^2 v}}+o(\rho^{\frac{\alpha}{2}}),\ \alpha<2,\\ \rho\cdot\dfrac{1-\sqrt{1+4sv}}{av}+o(\rho),\ \alpha=2,\\ -\dfrac{s\rho^{\alpha-1}}{a}+o(\rho^{\alpha-1}),\ \alpha>2,\end{cases}$$

where

$$v=\frac{a(p_1\beta_{12}+p_2\beta_{22})}{2}+\frac{1}{a(1-p)}\left(a^2\sum_{i=1}^{N}\frac{c_j}{a_j^2}-1\right),$$

$z(s)=p_1z_1(s)+p_2z_2(s)$ *is the solution of equation*
$p_1z_1+p_2z_2=p_1\beta_1(s-\mu_1(p_1z_1+p_2z_2))+p_2\beta_2(s-\mu_1(p_1z_1+p_2z_2))$.

Proof. Using assumption I and Lemma 2 it is possible to write

$$z(s\rho^\alpha)=1-(s\rho^\alpha-\mu_1(z(s\rho^\alpha)))\cdot\beta_1+(s\rho^\alpha-\mu_1(z(s\rho^\alpha)))^2\cdot\frac{\beta_2}{2}+o((s\rho^\alpha-\mu_1(z(s\rho^\alpha)))^2),\quad (3)$$

where $\beta_i=p_1\beta_{1i}+p_2\beta_{2i},\ i=1,2$.

Also, we may write next expansion for function $\mu_1(p_1z_1+p_2z_2)$:

$$\mu_1(z(s\rho^\alpha))=\mu_1^{'}(1)(z(s\rho^\alpha)-1)+\frac{\mu_1^{''}(1)}{2}(z(s\rho^\alpha)-1)^2+o((z(s\rho^\alpha)-1)^2).\quad (4)$$

Substitute (3) in (4), after easy manipulations quadratic equation for $z(s\rho^\alpha)-1$ is obtained:

$$av\cdot(z(s\rho^\alpha)-1)^2-\rho\cdot(z(s\rho^\alpha)-1)-\beta_1\cdot s\rho^\alpha+o(\max((z(s\rho^\alpha)-1)^2,\rho\cdot(z(s\rho^\alpha)-1),\rho^\alpha))=0,$$

its solutions:

$$z(s\rho^\alpha)-1=\frac{\rho\pm\sqrt{\rho^2+4\,s\rho^\alpha v}}{2av}+o(z(s\rho^\alpha)-1).$$

from here asymptotics in the lemma statement are obtained.

Corollary 1. Asymptotic expansion for $\mu_1^*(s\rho^\alpha)$ is:

$$\mu_1^*(s\rho^\alpha)=\begin{cases}-\sqrt{\dfrac{s\rho^\alpha}{v}}+o(\rho^{\frac{\alpha}{2}}),\ \alpha<2,\\ -\rho\cdot\dfrac{2\,s}{1+\sqrt{1+4sv}}+o(\rho),\ \alpha=2,\\ -s\rho^{\alpha-1}+o(\rho^{\alpha-1}),\ \alpha>2.\end{cases}$$

This expansion is obtained directly from Lemma 4 and (4)

Lemma 5. *The next asymptotics for* $p_{0j}(s\rho^\alpha)$ *are true :*

$$p_{0j}(s\rho^\alpha) = \begin{cases} \kappa_j \cdot \sqrt{\frac{v}{s}} \cdot \rho^{-\frac{\alpha}{2}} + o(\rho^{-\frac{\alpha}{2}}), \ \alpha < 2, \\ \kappa_j \cdot \frac{1+\sqrt{1+4sv}}{2\,s} \cdot \rho^{-1} + o(\rho^{-1}), \ \alpha = 2, \\ \kappa_j \frac{\rho^{1-\alpha}}{s} \cdot + o(\rho^{1-\alpha}), \ \alpha > 2, \end{cases}$$

where

$$\kappa_j = \prod_{n \neq j} \frac{a_n}{a_n - a_j} \cdot \prod_{k=2}^{N} \frac{\mu_k^*(0) + a_j(1 - p(p, z_k^*(0)))}{\mu_k^*(0)}.$$

Proof. Since, only $\mu_1(1) = 0$ then from (2):

$$\rho^\alpha p_{0j}(s\rho^\alpha) = \frac{\rho^\alpha}{s\rho^\alpha - \mu_1^*(s\rho^\alpha)} \times \prod_{n \neq j} \frac{a_n}{a_n - a_j} \cdot \prod_{k=2}^{N} \frac{\mu_k^*(0) + a_j(1 - p(p, z_k^*(0)))}{\mu_k^*(0)}.$$

Therefore and from corollary 1, lemma statement is obtained.

Lemma 6. *The next asymptotics for* $\mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma})$ *are true:*

$$\mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma}) = \frac{ap_2 u\rho^\gamma}{ap_1\beta_{11} - 1} + \psi\rho^{2\gamma} + o(\rho^{2\gamma}), \ \textit{where}$$

$$\psi = \begin{cases} \frac{ap_1\beta_{11}}{ap_1\beta_{11} - 1} s + \frac{ap_2 u^2}{2(1 - ap_1\beta_{11})} + \frac{\mu_1''(1)}{2} \frac{p_2^2 u^2}{(1 - ap_1\beta_{11})^3} + \frac{p_1\beta_{12} a^3 p_2^2 u^2}{2(1 - ap_1\beta_{11})^3}, \ \alpha \leqslant 2, \\ \frac{ap_2 u^2}{2(1 - ap_1\beta_{11})} + \frac{\mu_1''(1)}{2} \frac{p_2^2 u^2}{(1 - ap_1\beta_{11})^3} + \frac{p_1\beta_{12} a^3 p_2^2 u^2}{2(1 - ap_1\beta_{11})^3}, \ \alpha > 2. \end{cases}$$

Proof. Since $p_1 z_1^* + p_2 e^{-u\rho^\gamma} = p_1\beta_1(s\rho^\alpha - \mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma})) + p_2 e^{-u\rho^\gamma}$.
Denote: $\tau = p_1 z_1^* + p_2 e^{-u\rho^\gamma}$. Using the assumption II we have

$$\tau = p_1 \left(1 - \beta_{11}(s\rho^\alpha - \mu_1(\tau)) + \frac{\beta_{12}}{2}((s\rho^\alpha - \mu_1(\tau)))^2\right) + \\ + p_2 \left(1 - u\rho^\gamma + \frac{u^2\rho^{2\gamma}}{2}\right) + o(\rho^{2\gamma}). \quad (5)$$

Using the asymptotics for $\mu_1(\tau)$ and separate the principal part with degree ρ^γ we have

$$\tau - 1 = \frac{p_2 u\rho^\gamma}{ap_1\beta_{11} - 1} + + \psi\rho^{2\gamma} + o(\rho^{2\gamma}). \quad (6)$$

Substitute (6) in (5), ψ might be found.

Lemma 7. *The next asymptotics are true*:

$$\sum_{j=1}^{N} f_j(z_1^*, e^{-u\rho^\gamma}, s\rho^\alpha)\frac{c_j a_j}{\mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma}) + a_j(1 - p(p_1 z_1^* + p_2 e^{-u\rho^\gamma}))} =$$

$$= \frac{1}{1-p} \times \begin{cases} 1 + \dfrac{ap_2 u}{ap_1\beta_{11} - 1}\sqrt{\dfrac{v}{s}} + o(1), \ \alpha < 2, \\ 1 + \dfrac{ap_2 u}{ap_1\beta_{11} - 1}\dfrac{1+\sqrt{1+4sv}}{2\,s} + o(1), \ \alpha = 2, \\ 1 + \dfrac{ap_2 u\rho^{2-\alpha}}{ap_1\beta_{11} - 1}\dfrac{1}{s} + o(\rho^{2-\alpha}), \ \alpha > 2. \end{cases}$$

Proof. Using the definition of $f_j(z_1, z_2, s), \ j = \overline{1, N}$, the investigated expression might be rewritten in the next form:

$$\sum_{j=1}^{N} f_j(z_1^*, e^{-u\rho^\gamma}, s\rho^\alpha)\frac{c_j a_j}{\mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma}) + a_j(1 - p(p_1 z_1^* + p_2 e^{-u\rho^\gamma}))} =$$

$$= \frac{1}{(1-p)(p_1 z_1^* + p_2 e^{-u\rho^\gamma})} - (s\rho^\alpha - \mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma}))\times$$

$$\times \sum_{j=1}^{N} \frac{a_j p_{0j}(s)}{\mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma}) + a_j(1 - p(p_1 z_1^* + p_2 e^{-u\rho^\gamma}))} + o(\rho^{2\gamma}).$$

After using Lemma 5 and the fact that $\sum\limits_{j=1}^{N} \kappa_j = 1$, the lemma statement is obtained.

Lemma 8. *The next asymptotics are true*:

$$e^{-u\rho^\gamma} - \beta_2(s\rho^\alpha - \mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma})) =$$

$$\begin{cases} \dfrac{\beta_{21}\rho^{2\gamma}}{ap_1\beta_{11} - 1} \times \left[-s + \dfrac{a^2 p_2^2 v}{(ap_1\beta_{11} - 1)^2} u^2 \right] + o(\rho^{2\gamma}), \ \alpha < 2, \\ \dfrac{\rho^2}{ap_1\beta_{11} - 1} \times \left[u - \beta_{21} s + \dfrac{a^2 p_2^2 \beta_{21} v}{(ap_1\beta_{11} - 1)^2} u^2 \right] + o(\rho^2), \ \alpha = 2, \\ \dfrac{\rho^2}{ap_1\beta_{11} - 1} \times \left[u + \dfrac{a^2 p_2^2 \beta_{21} v}{(ap_1\beta_{11} - 1)^2} u^2 \right] + o(\rho^2), \ \alpha > 2. \end{cases}$$

Proof. These expressions might be obtained directly from Lemma 6 and the following expansion

$$e^{-u\rho^\gamma} - \beta_2(s\rho^\alpha - \mu_1(p_1 z_1^* + p_2 e^{-u\rho^\gamma})) =$$

$$= 1 - u\rho^\gamma + \frac{u^2\rho^{2\gamma}}{2} - 1 + \beta_{21}\left[s\rho^\alpha - \frac{ap_2 u\rho^\gamma}{ap_1\beta_{11} - 1} - \psi\rho^{2\gamma} \right] - \frac{\beta_{22}}{2}\left[\frac{ap_2 u\rho^\gamma}{ap_1\beta_{11} - 1} \right]^2 + o(\rho^{2\gamma}).$$

4.2 Main Theorem

Theorem 1. *While $m \to \infty$ the next limit exists*

$$\lim_{m\to\infty} \mathbb{P}\left(\rho^{\gamma} \cdot L_2\left(\frac{t}{\rho^{\alpha}}\right) < x\right) =$$

$$= \begin{cases} \sqrt{\frac{2}{\pi}} \cdot \int\limits_{0}^{\sqrt{\frac{v^*}{2t}}wx} e^{-\frac{y^2}{2}}\,dy, \ \alpha < 2, \\ 1 - \frac{e^{-wx}}{\sqrt{\pi}} \int\limits_{-\sqrt{\frac{t}{4v^*}}+wx\sqrt{\frac{v^*}{4t}}}^{+\infty} e^{-y^2}\,dy - \frac{1}{\sqrt{\pi}} \int\limits_{\sqrt{\frac{t}{4v^*}}+wx\sqrt{\frac{v^*}{4t}}}^{+\infty} e^{-y^2}\,dy, \ \alpha = 2, \\ 1 - e^{-wx}, \ \alpha > 2, \end{cases}$$

where

$$w = \frac{1 - a^* p_1^* \beta_{11}^*}{a^* p_2^* v^*}.$$

Proof. Instead of finding $\lim\limits_{m\to\infty} \mathbb{P}\left(\rho^{\gamma} \cdot L_2\left(\frac{t}{\rho^{\alpha}}\right) < x\right)$, we will find $\lim\limits_{\rho\to 0} \rho^{\alpha} \cdot p(1, e^{-u\rho^{\gamma}}, s\rho^{\alpha})$ and then inverse the Laplace transform to obtain the original limit distribution.
Using the expression from Lemma 3, the investigated limit might be rewritten in the next form:

$$\lim_{\rho\to 0}\Bigg\{\rho^{\alpha}\cdot p_0(s\rho^{\alpha}) + \rho^{\alpha}\cdot p_2(e^{-u\rho^{\gamma}}-1)\cdot\frac{1}{\mu_1(p_1+p_2e^{-u\rho^{\gamma}})(s\rho^{\alpha}-\mu_1(p_1+p_2e^{-u\rho^{\gamma}}))}\times$$
$$\times\Bigg[\gamma_2^{(1)}(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})\frac{1-e^{-u\rho^{\gamma}}}{(1-p)e^{-u\rho^{\gamma}}}\beta_2(s\rho^{\alpha}-\mu_1(p_1+p_2e^{-u\rho^{\gamma}}))+$$
$$+\frac{\prod\limits_{m=1}^{N}[\mu_1(p_1+p_2e^{-u\rho^{\gamma}})+a_m(1-p(p_1+p_2e^{-u\rho^{\gamma}}))]}{\alpha_1(p_1+p_2e^{-u\rho^{\gamma}})}\times$$
$$\times\sum_{j=1}^{N}\frac{c_j a_j f_j(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{\mu_1(p_1+p_2e^{-u\rho^{\gamma}})+a_j(1-p(p_1+p_2e^{-u\rho^{\gamma}}))}\Bigg]\Bigg\}.$$

Considering the fact, that $\lim\limits_{\rho\to 0}\rho^{\alpha}p_0(s\rho^{\alpha}) \to 0$ from lemma 5, $\mu_1(p_1+p_2e^{-u\rho^{\gamma}}) = ap_2u\rho^{\gamma} + o(\rho^{\gamma})$. From Lemma 1 it is possible to find

$$\alpha_1(1) = \prod_{m=2}^{N}(-\mu_m(1)) = \frac{1}{a(1-p)}\prod_{m=1}^{N}(a_m(1-p)),$$

then

$$\lim_{\rho\to 0}\frac{\prod\limits_{m=1}^{N}[\mu_1(p_1+p_2e^{-u\rho^{\gamma}})+a_m(1-p(p_1+p_2e^{-u\rho^{\gamma}}))]}{\alpha_1(p_1+p_2e^{-u\rho^{\gamma}})} = \frac{\prod\limits_{m=1}^{N}(a_m(1-p))}{\alpha_1(1)} = a(1-p).$$

So that, the task is equal to finding the next limit:

$$\lim_{\rho\to 0}\left\{\frac{\rho^{\alpha}}{a^2p_2(1-p)(p_1+p_2e^{-u\rho^{\gamma}})}\gamma_2^{(1)}(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})+\right.$$
$$\left.+\frac{\prod_{m=1}^{N}(a_m(1-p))}{\prod_{m=2}^{N}(-\mu_m(1))}\frac{\rho^{\alpha-\gamma}}{a^2p_2u}\sum_{j=1}^{N}\frac{c_ja_jf_j(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{\mu_1(p_1+p_2e^{-u\rho^{\gamma}})+a_j(1-p(p_1+p_2e^{-u\rho^{\gamma}}))}\right\}.$$

Similary to the proof of the Lemma 7, we have

$$\lim_{\rho\to 0}\frac{\rho^{\alpha-\gamma}}{a^2p_2u}\sum_{j=1}^{N}f_j(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})\frac{c_ja_j}{\mu_1(p_1+p_2e^{-u\rho^{\gamma}})+a_j(1-p(p_1+p_2e^{-u\rho^{\gamma}}))}=0\ \ \forall\alpha>0.$$

Therefore,

$$\lim_{\rho\to 0}\rho^{\alpha}p(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})=\lim_{\rho\to 0}\frac{\rho^{\alpha}\gamma_2^{(1)}(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{a^2p_2(1-p)(p_1+p_2e^{-u\rho^{\gamma}})}=$$
$$=\lim_{\rho\to 0}\frac{\prod_{m=1}^{N}(a_m(1-p))}{\alpha_1(1)(1-p)a^2p_2}\sum_{e=1}^{N}\rho^{\alpha}p_{2e}(e^{-u\rho^{\gamma}},0,s\rho^{\alpha})=$$
$$=\lim_{\rho\to 0}\frac{1}{ap_2}\sum_{e=1}^{N}\rho^{\alpha}p_{2e}(e^{-u\rho^{\gamma}},0,s\rho^{\alpha})=\lim_{\rho\to 0}\frac{1}{p_2a^2(1-p)}\rho^{\alpha}\gamma_2^{(1)}(z_1^*,e^{-u\rho^{\gamma}},s\rho^{\alpha}).$$

From the definition of $\gamma_2^{(1)}(z_1,z_2,s)$, we have

$$\lim_{\rho\to 0}\frac{\rho^{\alpha}\gamma_2^{(1)}(z_1^*,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{p_2a^2(1-p)}=$$
$$=\lim_{\rho\to 0}\frac{(p_1z_1^*+p_2e^{-u\rho^{\gamma}})e^{-u\rho^{\gamma}}\cdot\rho^{\alpha}}{p_2a^2\alpha_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}})\cdot[e^{-u\rho^{\gamma}}-\beta_2(s\rho^{\alpha}-\mu_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}}))]}\times$$
$$\times\prod_{m=1}^{N}[\mu_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}})+a_m(1-p(p_1z_1^*+p_2e^{-u\rho^{\gamma}}))]\times$$
$$\times\sum_{j=1}^{N}\frac{c_ja_jf_j(z_1^*,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{\mu_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}})+a_j(1-p(p_1z_1^*+p_2e^{-u\rho^{\gamma}}))}=$$
$$=\lim_{\rho\to 0}\frac{(1-p)\rho^{\alpha}}{p_2a[e^{-u\rho^{\gamma}}-\beta_2(s\rho^{\alpha}-\mu_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}}))]}\times$$
$$\times\sum_{j=1}^{N}\frac{c_ja_jf_j(z_1^*,e^{-u\rho^{\gamma}},s\rho^{\alpha})}{\mu_1(p_1z_1^*+p_2e^{-u\rho^{\gamma}})+a_j(1-p(p_1z_1^*+p_2e^{-u\rho^{\gamma}}))}.$$

Using Lemmas 7 and 8, we have

$$\lim_{\rho\to 0}\rho^{\alpha}p(1,e^{-u\rho^{\gamma}},s\rho^{\alpha})=\begin{cases}\left[s\cdot\left(1+\frac{a^*p_2^*u}{1-a^*p_1^*\beta_{11}^*}\sqrt{\frac{v^*}{s}}\right)\right]^{-1}, & \alpha<2,\\ \left[s\cdot\left(1+\frac{a^*p_2^*u}{1-a^*p_1^*\beta_{11}^*}\cdot\frac{2v^*}{1+\sqrt{1+4sv^*}}\right)\right]^{-1}, & \alpha=2,\\ \left[s\cdot\left(1+\frac{a^*p_2^*v^*}{1-a^*p_1^*\beta_{11}^*}u\right)\right]^{-1}, & \alpha>2.\end{cases}$$

After inversing the Laplace transform, the theorem statement is obtained.

Corollary 2. *Probability density function of the limit distribution is:*

$$\begin{cases}\sqrt{\frac{v^*}{\pi t}}w\cdot exp\left\{-\frac{v^*w^2x^2}{4t}\right\}, & \alpha<2,\\ \frac{we^{-wx}}{\sqrt{\pi}}\int\limits_{-\sqrt{\frac{t}{4v^*}}+wx\sqrt{\frac{v^*}{4t}}}^{+\infty}e^{-y^2}dy+w\sqrt{\frac{v^*}{4\pi t}}\times \\ \times\left(exp\left\{-wx-\left(-\sqrt{\frac{t}{4v^*}}+wx\sqrt{\frac{v^*}{4t}}\right)^2\right\}+exp\left\{\left(\sqrt{\frac{t}{4v^*}}+wx\sqrt{\frac{v^*}{4t}}\right)^2\right\}\right), & \alpha=2,\\ w\cdot exp\{-wx\}, & \alpha>2.\end{cases}$$

Corollary 3. *Mathematical expectation of the limit distribution is equal to*

$$\sqrt{\frac{t}{v^*\pi}}\cdot\frac{2}{w},\ if\ \alpha<2,$$

and

$$\frac{1}{w},\ if\ \alpha>2.$$

Corollary 4. *Variance of the limit distribution is equal to*

$$\frac{(2\pi-4)t}{\pi v^*w^2},\ if\ \alpha<2$$

and

$$\frac{1}{w^2},\ if\ \alpha>2.$$

Remark 1. Mathematical expectation and variance in case $\alpha=2$ should be calculated using numerical methods.

5 Numerical Examples

For visualisation results of this paper let us consider the system with parameters: $n=2,a_1=1,a_2=2,c_1=0.35,c_2=0.65,\beta_{11}=0.5749,\beta_{21}=0.775,\beta_{12}=1,\beta_{22}=1,p=0.5,p_1=0.5,p_2=0.5$. The plots below show the density functions for all different cases for α and for different parameter t.

Also let us show, how mathematical expectation is changing, while parameters p_2 or p are being changed, the main parameters of the system are $n=2,a_1=1,a_2,c_1=0.2,c_2=0.8$, the other parameters has been chosen such way that ρ is close to 0 (Figs. 1, 2, 3 and Tables 1, 2).

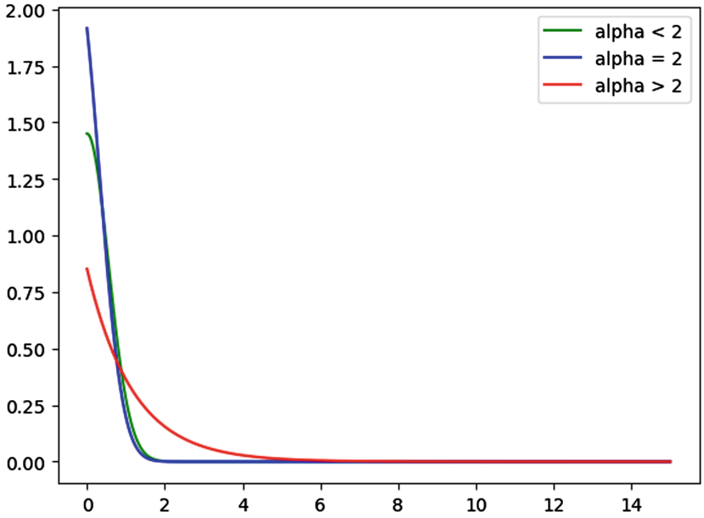

Fig. 1. t = 0.1

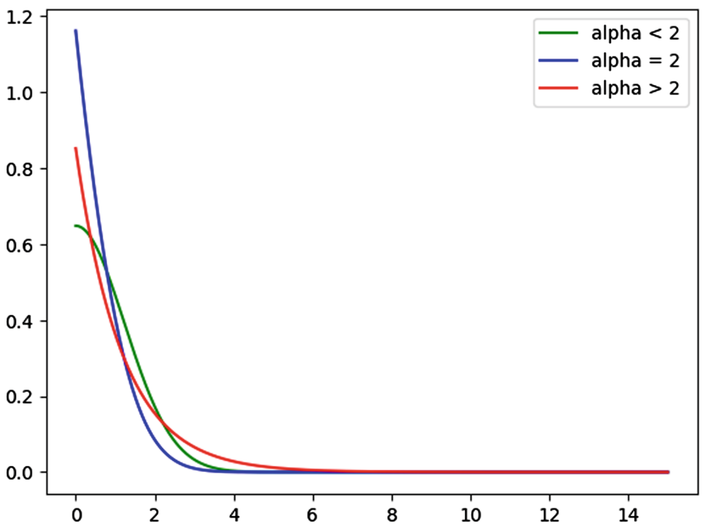

Fig. 2. t = 0.5

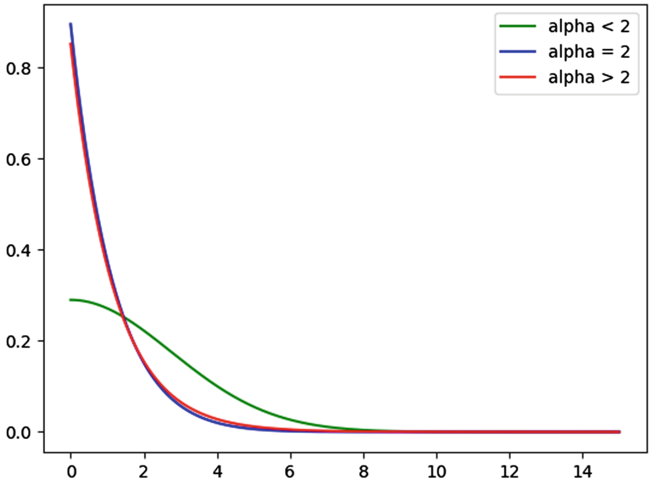

Fig. 3. t = 2.5

Table 1. Mathematical expectation while the probability of going to the lowest class is being changed

p_2	$t = 0.1$			$t = 1$			$t = 10$		
	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$
0.05	0.595	0.522	1.802	1.882	1.242	1.802	5.952	1.785	1.802
0.25	0.605	0.53	1.83	1.912	1.261	1.83	6.046	1.813	1.83
0.5	0.617	0.541	1.867	1.95	1.286	1.867	6.167	1.849	1.867
0.75	0.629	0.552	1.905	1.99	1.313	1.905	6.292	1.887	1.905
0.95	0.64	0.561	1.936	2.023	1.334	1.936	6.397	1.918	1.936

Table 2. Mathematical expectation while the probability of repeating intensity is being changed:

p	$t = 0.1$			$t = 1$			$t = 10$		
	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$	$\alpha < 2$	$\alpha = 2$	$\alpha > 2$
0.05	0.561	0.483	1.493	1.106	1.493	1.493	1.488	1.493	1.493
0.25	0.566	0.489	1.524	1.122	1.524	1.524	1.518	1.524	1.524
0.5	0.58	0.502	1.598	1.161	1.598	1.598	1.59	1.598	1.598
0.75	0.619	0.541	1.818	1.275	1.818	1.818	1.804	1.818	1.818
0.95	0.868	0.789	3.581	2.023	3.581	3.581	3.425	3.581	3.581

6 Conclusion

The main result of this paper is an explicit form of the limit distribution of the queue length for the least priority class, which has been obtained. For each cases ($\alpha < 2,\ \alpha = 2$ and $\alpha > 2$) expressions for the density function are obtained. For cases ($\alpha < 2$ and $\alpha > 2$) mathematical expectation and variance are given in explicit form, in case $\alpha = 2$ these characteristics might be calculated numerically. Numerical examples show us necessity to consider parameter t, since relation among considered cases is changing while t is being changed. Provided theoretical results of this article can be used to analyse real queueing systems in which there is a correlation of the intervals between customer arrivals.

References

1. Bergovin, A.K., Ushakov, V.G.: Discipline-priority queuing systems without serving interruptions. Moscow Univ. Comput. Math. Cybern. **42**(3), 119–125 (2018). https://doi.org/10.3103/S0278641918030032
2. Gusella, R.: Characterizing the variability of arrival processes with indexes of dispersion. IEEE J. Sel. Areas Commun. **2**, 203–211 (1991). https://doi.org/10.1109/49.68448
3. Paxson, V., Floyd, S.: Wide-Area Traffic: the failure of poisson modelling. IEEE/ACM Trans. Netw. (ToN) **3**, 226–244 (1995). https://doi.org/10.1109/90.392383
4. Hwang, G.U., Choi, B.D., Kim, J.-K.: The waiting time analysis of a discrete-time queue with arrivals as an autoregressive process of order 1. J. Appl. Probab. **3**, 619–629 (2002). https://doi.org/10.1239/jap/1034082132
5. Kamoun, F.: The discrete-time queue with autoregressive inputs revisited. Queueing Syst., 185–192 (2006). https://doi.org/10.1007/s11134-006-9591-3
6. Prokhorov, Y.V.: Transitional phenomena in queuing processes. Lith. Math. Collection **3**(1), 199–206 (1963)
7. Kingman, J.F.C.: On queues in heavy traffic. J. Rey. Stat. Soc. B-25, 383–392 (1962). https://doi.org/10.1111/j.2517-6161.1962.tb00465.x
8. Borovkov, A.A.: Asymptotic methods in queuing theory. M., Nauka, 357 (1979)
9. Nazarov, A.A., Moiseeva, S.P.: Method of Asymptotic Analysis in Queuing Theory, p. 112. Tomsk state University, Tomsk (2006)
10. Garayshina, I.R., Moiseeva, S.P., Nazarov, A.A.: Methods for studying correlated flows and special queuing systems Tomsk: NTL Publishing House, p. 206 (2010)
11. Nazarov, A., Paul, S., Lizyura, O.: Asymptotic analysis of Markovian retrial queue with unreliable server and multiple types of outgoing calls. Global Stoch. Anal. **8**(3), 143–149 (2021)
12. Danielyan, E.A.: Description of one class of limit distributions in single-channel priority systems, pp. 48–52. VNIISI, Queuing theory. M. (1981)
13. Ushakov, A.V.: Heavy-traffic analysis for the queueing system with hyper exponential stream. Inform. Appl. **6**(3), 117–121 (2012)
14. Hooke, J.A.: Some heavy-traffic limit theorems for a priority queue with general arrivals. Operat. Res. **20**, 381–388 (1972). https://doi.org/10.1287/opre.20.2.381
15. Hooke, J.A., Prabhy, N.V.: Priority queues in heavy traffic. Opns. Res., 1–9 (1971)
16. Abate, J., Whitt, W.: Asymptotics for M—G—1 low-priority waiting-time tail probabilities. Queueing Syst. **25**, 173–233 (1997)