Maximum Principle and Necessary Conditions for an Optimal Control Problem Governed by a Parabolic Equation

Irina Astashova[1,2,a)], Alexey Filinovskiy[3,4,b)] and Dmitriy Lashin[5,c)]

[1]*Lomonosov Moscow State University, Department of Mathematics and Mechanics, Leninskie Gory, 1, 119991, Moscow, Russia*
[2]*Plekhanov Russian University of Economics,Stremyanny lane, 36, 117997, Moscow, Russia*
[3]*Bauman Moscow State Technical University, 2nd Baumanskaya str., 5, 105005, Moscow, Russia*
[4]*Lomonosov Moscow State University, Department of Mathematics and Mechanics, Leninskie Gory, 1, 119991, Moscow, Russia*
[5]*FITO Research and Production Company, Kaluzhskoe shosse, 23 km, 14-3, Moscow, Russia*

[a)]Corresponding author: ast.diffiety@gmail.com
[b)]flnv@yandex.ru
[c)]dalashin@gmail.com

Abstract. We prove maximum principle for a parabolic equation and obtain necessary conditions for an optimal control problem governed by a one-dimensional general type parabolic equation. We obtain lower estimates to the control function without assumption of the data positivity.

Introduction. Definitions and Preliminary Results.

We study the problem which is modeled by a one-dimensional parabolic equation. First studies of this problem are presented in [1]–[3]. The mathematical approach is developed in [4] and [5]. Various extremum problems for final and distributed observation are considered in [6]—[13]. One can find the review of early results in [10], later results is contained in [14], [15].

We consider the extremum problem with weighted integral cost functional for the following parabolic mixed problem

$$u_t = (a(x,t)u_x)_x + b(x,t)u_x + h(x,t)u, \qquad (x,t) \in Q_T = (0,1)\times(0,T), \quad T > 0, \tag{1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \qquad 0 < t < T, \tag{2}$$

$$u(x,0) = \xi(x), \qquad 0 < x < 1, \tag{3}$$

where the real functions a, b and h are smooth in $\overline{Q}_T$,

$$0 < a_1 \le a(x,t) \le a_2 < \infty, \tag{4}$$

$\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$. Here $W_2^1(0,T)$ is the Sobolev space of weakly differentiable functions with the norm

$$\|u\|_{W_2^1(0,T)} = \left(\int_0^T ((u')^2 + u^2)dt\right)^{1/2}. \tag{5}$$

We study the control problem with point observation: by controlling the temperature φ at the left end of the segment (the functions ψ and ξ are assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z(t)$ over the entire time interval $(0,T)$. This problem arises in the model of climate control in industrial greenhouses [16]–[17]. The proposed paper develops and generalizes the authors' results of [16]–[22]. Here we study a more general equation with a variable diffusion coefficient a, a convection coefficient b, and a potential h called the depletion potential, and obtain estimates for the control function with the help of a maximum principle without assumption of the data positivity. We give also a necessary condition to minimizer.

Definition 1. ([24], p. 6.) We denote by $V_2^{1,0}(Q_T)$ the Banach space of functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0\le t\le T} \|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)} \tag{6}$$

such that $t \mapsto u(\cdot,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

We denote by $\widetilde{W}_2^1(Q_T)$ the set of functions $\eta \in W_2^1(Q_T)$ satisfying the conditions $\eta(x,T) = 0$ and $\eta(0,t) = 0$.

Definition 2. A *weak solution* to the problem (1)–(3) is a function $u \in V_2^{1,0}(Q_T)$ satisfying the condition $u(0,t) = \varphi(t)$ and the equality

$$\int_{Q_T} \Big(a(x,t)u_x\eta_x - b(x,t)u_x\eta - h(x,t)u\eta - u\eta_t\Big)\,dx\,dt = \int_0^1 \xi(x)\,\eta(x,0)\,dx + \int_0^T a(1,t)\,\psi(t)\,\eta(1,t)\,dt \tag{7}$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$.

Theorem 1. *([21]) The problem (1)–(3) has a unique weak solution $u \in V_2^{1,0}(Q_T)$, which satisfies the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C\big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)} + \|\xi\|_{L_2(0,1)}\big) \tag{8}$$

with some constant C independent of φ, ψ, and ξ.

Theorem 2. *Let u be a solution to the problem (1)–(3) with the boundary and initial functions φ, ψ, and ξ satisfying the conditions $\operatorname*{ess\,inf}_{t\in(0,T)} \varphi \ge 0$, $\operatorname*{ess\,inf}_{t\in(0,T)} \psi \ge 0$, and $\operatorname*{ess\,inf}_{x\in(0,1)} \xi \ge 0$. Then the solution u is also non-negative:*

$$\operatorname*{ess\,inf}_{(x,t)\in Q_T} u \ge 0. \tag{9}$$

Main Results

Continuous Dependence on the Data and Estimates of Solutions

Theorem 3. *The solution $u \in V_2^{1,0}(Q_T)$ to the problem (1)–(3) continuously depends on the initial and boundary data $(\xi,\varphi,\psi) \in L_2(0,1) \times W_2^1(0,T) \times W_2^1(0,T)$.*

Using Theorem 2, we obtain the following estimate. In comparison with our previous results, we obtain this result without assumption of the data positivity.

Theorem 4. *Let the functions a, b, and h satisfy the conditions $a_t \ge 0$, $b_x - h \ge 0$ on Q_T, $b \ge 0$ on $[0,x_0]\times[0,T]$ with $x_0 \in (0,1]$, and $b(1,t) \le 0$ for all $t \in [0,T]$. Then the solution u of the problem (1)–(3) satisfies the inequality*

$$\|u(x_0,t)\|_{L_1(0,T)} \le \|\varphi\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right). \tag{10}$$

Corollary 1. *Let x_0 and the functions a, b, h, satisfy the conditions of Theorem 4. If $\psi = 0$ and $\xi = 0$, then the solution to the problem (1)–(3) satisfies the following inequality:*

$$\|u(x_0,t)\|_{L_1(0,T)} \le \|\varphi\|_{L_1(0,T)}. \tag{11}$$

Extremum problem

We denote by $\Phi \subset W_2^1(0,T)$ the set of control functions φ and by $Z \subset L_2(0,T)$ the set of objective functions z. Further we suppose that Φ is a non-empty, closed, convex and bounded set. Consider the weighted integral cost functional

$$J[z,\rho,\varphi] = \int_0^T (u_\varphi(x_0,t) - z(t))^2 \rho(t)dt, \tag{12}$$

where $x_0 \in (0,1)$, $\varphi \in \Phi$, $z \in Z$, $u_\varphi \in V_2^{1,0}(Q_T)$ is the solution to the problem (1)–(3) with the given control function φ, and $\rho \in L_\infty(0,T)$ is a real-valued weight function with

$$0 < \rho_1 = \operatorname*{ess\,inf}_{t\in(0,T)} \rho(t). \tag{13}$$

Assuming the functions z and ρ to be fixed, consider the minimization problem

$$m[z,\rho,\Phi] = \inf_{\varphi\in\Phi} J[z,\rho,\varphi]. \tag{14}$$

In ([21]) the following result is obtained.

Theorem 5. *For any $z \in L_2(0,T)$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$m[z,\rho,\Phi] = J[z,\rho,\varphi_0].$$

Necessary condition for a minimizer

Theorem 6. *Let $\varphi_0 \in \Phi$ be a minimizer. Then for any $\varphi \in \Phi$ the following inequality holds:*

$$\int_0^T \left(u_{\varphi_0}(x_0,t) - z(t)\right)\left(u_\varphi(x_0,t) - u_{\varphi_0}(x_0,t)\right)\rho(t)dt \geq 0.$$

Lower Estimates for Control Function

Theorem 4 implies a lower estimate for the norm of the control function in terms of the value of the quality functional.

Theorem 7. *Let x_0 and the functions a, b, h, satisfy the conditions of Theorem 4. Then the following inequality holds:*

$$\|\varphi\|_{L_1(0,T)} \geq \|z\|_{L_1(0,T)} - \left(\frac{TJ[z,\rho,\varphi]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right). \tag{15}$$

Corollary 2. *Let x_0 and the functions a, b, h, satisfy the conditions of Theorem 4. Suppose $\psi = \xi = 0$. Then the following inequality holds:*

$$\|\varphi\|_{L_1(0,T)} \geq \|z\|_{L_1(0,T)} - \left(\frac{TJ[z,\rho,\varphi]}{\rho_1}\right)^{1/2}. \tag{16}$$

Proofs

Proof of Theorem 4 is based on the following lemma for non-negative φ, ψ, and ξ.

Lemma 1. *(See [23].) Let the functions a, b, h satisfy the conditions $a_t \geq 0$ and $b_x - h \geq 0$ on Q_T, $b \geq 0$ on $[0,x_0]\times[0,T]$ with $x_0 \in (0,1]$, and $b(1,t) \leq 0$ for all $t \in [0,T]$. Then any solution u to the problem (1)–(3) with $\varphi \geq 0$, $\psi \geq 0$, and $\xi \geq 0$ satisfies the inequality*

$$\|u(x_0,t)\|_{L_1(0,T)} \leq \|\varphi\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right). \tag{17}$$

Proof. To prove Theorem 4, note that any function f can be represented as the difference $f = f^+ - f^-$ with $f^+(x) = \{0, f(x)\} \geq 0$ and $f^-(x) = \{0, -f(x)\} \geq 0$. We have $\varphi^\pm \in W_2^1(0,T)$, $\psi^\pm \in W_2^1(0,T)$, and $\xi^\pm \in L_2(0,T)$. Considering the solutions u^+ and u^- to the problem (1)–(3) with the corresponding data functions $\varphi^\pm$, $\psi^\pm$, and $\xi^\pm$, we obtain from (17) the following estimates:

$$\|u^+(x_0,t)\|_{L_1(0,T)} \leq \|\varphi^+\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi^+\|_{L_1(0,T)} + \|\xi^+\|_{L_1(0,1)}\right), \tag{18}$$

$$\|u^-(x_0,t)\|_{L_1(0,T)} \leq \|\varphi^-\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi^-\|_{L_1(0,T)} + \|\xi^-\|_{L_1(0,1)}\right). \tag{19}$$

Therefore,

$$\begin{aligned}
&\|u(x_0,t)\|_{L_1(0,T)} \leq \|u^+(x_0,t)\|_{L_1(0,T)}\|\varphi\|_{L_1(0,T)} + \|u^-(x_0,t)\|_{L_1(0,T)} \\
&\leq \|\varphi^+\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi^+\|_{L_1(0,T)} + \|\xi^+\|_{L_1(0,1)}\right) \\
&+ \|\varphi^-\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi^-\|_{L_1(0,T)} + \|\xi^-\|_{L_1(0,1)}\right) \\
&= \|\varphi\|_{L_1(0,T)} + \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right).
\end{aligned}$$

□

Now we prove Theorem 6.

Proof. Denote by $\varphi_0 \in \Phi$ a minimizer for the problem (1)–(3), (14) (the minimizer exists by virtue of Theorem 5). Now, for arbitrary $\varphi \in \Phi$ we obtain, by the convexity of Φ, that $\varphi_0 + \gamma(\varphi - \varphi_0) \in \Phi$ for $\gamma \in [0,1]$. Then

$$\begin{aligned}
0 &\leq \frac{d}{d\gamma} J[z,\rho,\varphi_0 + \gamma(\varphi-\varphi_0)]\Big|_{\gamma=0} \\
&= \frac{d}{d\gamma}\int_0^T (u_{\varphi_0+\gamma(\varphi-\varphi_0)}(x_0,t) - z(t))^2\rho(t)dt\Big|_{\gamma=0} \\
&= 2\int_0^T (u_{\varphi_0+\gamma(\varphi-\varphi_0)}(x_0,t) - z(t))(u_\varphi(x_0,t) - u_{\varphi_0}(x_0,t))\rho(t)dt\Big|_{\gamma=0} \\
&= 2\int_0^T (u_{\varphi_0}(x_0,t) - z(t))(u_\varphi(x_0,t) - u_{\varphi_0}(x_0,t))\rho(t)dt,
\end{aligned} \tag{20}$$

and Theorem 6 is proved. □

To prove Theorem 7 we need the following result.

Lemma 2. *Let B be a Banach space with the norm $\|\cdot\|_B$. Then for all $g \in B$ and $y \in B$ we have*

$$\|g\|_B \geq \|y\|_B - \|y-g\|_B. \tag{21}$$

Proof. The triangle inequality $\|p+q\|_B \leq \|p\|_B + \|q\|_B$ with $p = g$ and $q = y - g$ yields

$$\|y\|_B \leq \|g\|_B + \|y-g\|_B. \tag{22}$$

Now, from (22) we obtain the inequality (21). □

Proof. To prove Theorem 7, applying Lemma 2 for $B = L_1(0,T)$, $g = u(x_0,\cdot)$, and $y = z$, we get

$$\|u_\varphi(x_0,\cdot)\|_{L_1(0,T)} \geq \|z(\cdot)\|_{L_1(0,T)} - \|u_\varphi(x_0,\cdot) - z(\cdot)\|_{L_1(0,T)}. \tag{23}$$

Now, it follows from the Holder inequality and (13) that

$$\begin{aligned}
&\|u_\varphi(x_0,\cdot) - z(\cdot)\|_{L_1(0,T)} \\
&\leq T^{1/2}\|u_\varphi(x_0,\cdot) - z(\cdot)\|_{L_2(0,T)} \leq \left(\frac{TJ[z,\rho,\varphi]}{\rho_1}\right)^{1/2}.
\end{aligned} \tag{24}$$

By (10) and (24) we obtain

$$\begin{aligned}\|\varphi\|_{L_1(0,T)} &\geq \|u_\varphi(x_0,\cdot)\|_{L_1(0,T)} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right) \\ &\geq \|z(t)\|_{L_1(0,T)} - \left(\frac{TJ[z,\rho,\varphi]}{\rho_1}\right)^{1/2} - \frac{x_0}{a_1}\left(a_2\|\psi\|_{L_1(0,T)} + \|\xi\|_{L_1(0,1)}\right).\end{aligned} \tag{25}$$

□

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