Generalized Chaplygin problem

Yuri Kiselev[1], **Sergey Avvakumov**[2]

[1] *Lomonosov Moscow State University, Moscow, Russia;* `kiselev@cs.msu.su`
[2] *Lomonosov Moscow State University, Moscow, Russia;* `asn@cs.msu.su`

The following two-dimensional optimal control problem is studied:

$$
\begin{cases}
\dot{x} = u; \quad x \in I\!R^2, \quad u \in U \subset I\!R^2; \quad 0 \le t \le T; \\
x(0) = x(T) = a \in I\!R^2, \quad \dot{x}(0)/\|\dot{x}(0)\| = q(\alpha_0); \\
L[u] \equiv \displaystyle\int_0^T \big(A^* x, u\big)\, dt \ \longrightarrow \max_{u(\cdot)} .
\end{cases}
\tag{1}
$$

Here $q(\alpha) = \left(\begin{smallmatrix} \cos\alpha \\ \sin\alpha \end{smallmatrix}\right)$ is a unit vector, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the (2×2)-matrix, A^* is the transposed matrix A. The collection of known parameters T, U, a, $\alpha_0 \in [0, 2\pi)$ is given. The control domain U is a plane smooth convex compact set, $0 \in \operatorname{int} U$; support function $c(\psi) = \max\limits_{u \in U}(u, \psi)$ of the set U and its distant functions play the important technical role for problem (1) solution. It is assumed that $c(\psi) > 0 \ \forall \psi \neq 0$, gradient $c'(\psi)$ and Hessian $c''(\psi)$ are defined and continuous for all $\psi \neq 0$, $\operatorname{rank} c''(\psi) = 1$ for $\psi \neq 0$, [2]. Geometric sense of the cost L is doubled area of the plane figure, which is limited by the closed curve $x = x(t)$, $0 \le t \le T$. In the original Chaplygin's problem [1] about maximal flyby area (single anticlockwise bypass) the control domain U is a circle with translated center, $0 \in \operatorname{int} U$.

The Pontryagin maximum principle is used for solving the problem (1). In the study of the maximum principle boundary value problem

$$
\begin{cases}
\dot{x} = \quad c'(A^* x + \psi), \quad x\big|_{t=0} = x\big|_{t=T} = a, \quad \dot{x}(0)/\|\dot{x}(0)\| = q(\alpha_0), \\
\dot{\psi} = A^* c'(A^* x + \psi),
\end{cases}
$$

the theorem about support function's gradient and the vector first integral $A^* x - \psi = \text{const}$ are used. The important role belongs to the following two-dimensional Hamiltonian system

$$
\dot{p} = A^* c'(p), \quad p\big|_{t=0} = p^0, \quad p, p^0 \in I\!R^2 \setminus \{0\}
$$

analysis. Solution of this system admits certain description in an analytical form.

In the formulation of final result the polar set $\tilde{U}$ of the convex compact set U is involved. Optimal motion is running along the curve obtained from the polar curve $\partial\tilde{U}$ as result of some linear transformations depending on the parameters of the problem (1).

Support function's technique is useful in construction of numerical algorithms for optimal solution search and also in theoretical investigations [3-10]. In conclusion it is possible to draw attention to short publication [4] concerning Chaplygin's problem.

The authors were supported by the RFBR (no. 12-01-00175) and PSLSS (no. 65590.2010.1).

REFERENCES

1. *M.A. Lavrent'ev, L.A. Lyusternik.* A course in variational calculus, ONTI, Moscow-Leningrad (1938) (In Russian).
2. *Yu.N. Kiselev, S.N. Avvakumov, M.V. Orlov.* Optimal control. Linear theory and applications, MAKS Press, Moscow (2007) (In Russian).
3. *Yu.N. Kiselev.* "Generalized Chaplygin's problem," Proceedings of the Tenth Crimean Autumn Mathematical School, **10**, 160–163 (2000).
4. *Yu.N. Kiselev.* "Optimal design in a smooth linear time-optimal problem," Differential Equations, **26**, No. 2, 174–178 (1990).
5. *S.N. Avvakumov, Yu.N. Kiselev, M.V. Orlov.* "Methods for solving optimal control problems, based on Pontryagin's maximum principle," Trudy Mat. Inst. Steklov, Optim. Upr. i Differ. Uravn., **211**, 174–178 (1995).
6. *Yu.N. Kiselev.* "Rapidly converging algorithms for solving time-optimality problem," Kibernetika (Kiev), No. 6, 47–57, 62 (1990) (In Russian).
7. *Yu.N. Kiselev, M.V. Orlov.* "Numerical algorithms for linear time-optimalities," Zh. Vychisl. Mat. i Mat. Phys., **31**, No. 12, 39–43 (1991) (In Russian).
8. *S.N. Avvakumov, Yu.N. Kiselev.* "Support functions of some special sets, constructive smoothing procedures, and geometric differences," Problems in dynamic control; Mosk. Gos. Univ. im Lomonosova, Fak. VMK, Moscow, No. 1, 24–110 (2005) (In Russian).
9. *S.N. Avvakumov, Yu.N. Kiselev.* "Some algorithms of optimal control," Proc. Steklov Ins. Math, Control; Stability, and Inverse Problems of Dynamics, suppl. 2, Ekaterinburg, **12**, No. 2, 3–17 (2006) (In Russian).
10. *Yu.N. Kiselev.* "Construction of exact solutions for the nonlinear time-optimality problem of a special type," Fundam. Prikl. Math., **3**, No. 3, 847–868 (1997) (In Russian).