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On the Generation of Harmonics in Modes of Spatiotemporal Solitons

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An analytical study of the generation of spectral harmonics in the modes of formation of steady-state spatiotemporal solitons localized in all directions ("light bullets") is carried out. A model of the medium in which the optical degree of nonlinearity coincides with the number $N \geq 2$ of generated harmonics is offered for this purpose. It is shown that, within such a model the simultaneous generation of the harmonics and the formation of the light bullets are possible for $N = 2$ and $N \geq 6$. At the same time, it is necessary that both the basic frequency and its harmonics belong to the spectral range of the negative dispersion of the group velocity.

1. Introduction

Spatiotemporal solitons ("light bullets") and the conditions of their formation have been are extensively investigated in the last few decades.[1–21] However the processes of the formation of light bullets under the simultaneous generation of the highest harmonics are still insufficiently studied. Much attention has been paid to the formation of bullets under the generation of the second harmonics ($N = 2$). Both theoretical, and experimental works devoted to this question are known.[22–30] We also mention Ref. 31, in which the third-harmonic generation was observed in the mode of formation of the light bullet. It was established experimentally that the observed light bullet changed its shape nonperiodically in the process of evolution.[31] In Ref. 32, the formation of multi-colored "light bullets" was studied. In Ref. 33, the optical-terahertz bullets were studied. At the same time, the investigation of spatiotemporal solitons under the generation of arbitrary order N ($N = 2, 3, 4, \ldots$) harmonics is of considerable interest. This work is devoted to this study. The direct, but not cascade, modes of the generation of harmonics will be considered. The soliton will be understood not in strict mathematical interpretation here. The property of integrability of the relevant nonlinear systems of the equations is not obligatory. Below, we will call as spatiotemporal solitons the stable running light pulses, which are localized in all directions.

This article is organized as follows. In Sect. 2, a physical model is offered describing a direct (not cascade) process of the generation of the highest harmonic with any serial number. Here, a nonlinear system of two wave equations for the envelopes of the main frequency and its harmonic is derived. In Sect. 3, the one-dimensional soliton-like solutions of this system are constructed, when the conditions of phase and group matching are satisfied. In Sect. 4, by of an approximate method of averaged Lagrangian, the solutions in the form of three-dimensional spatiotemporal solitons are obtained. The corresponding numbers of harmonics are determined. The main conclusions of this work and further planned investigations are given in Sect. 5.

2. Physical Model

Let the pulses propagate along the z-axis. Then, we will write the wave equation as

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2} - \Delta_\perp E, \qquad (1)$$

where E and P are the electric field of the pulse and the polarization response of the medium, respectively, c is the speed of light in vacuum, t is time, and $\Delta_\perp$ is the transversal Laplacian.

The field E is represented as

$$E = \psi_1(z,t,\mathbf{r}_\perp)\exp[i(\omega t - k_1 z)] + \psi_N(z,t,\mathbf{r}_\perp)\exp[i(N\omega t - k_N z)] + \text{c.c.}, \qquad (2)$$

where $\psi_1(z,t,\mathbf{r}_\perp)$ and $\psi_N(z,t,\mathbf{r}_\perp)$ are the envelopes of the basic and highest harmonics, $\mathbf{r}_\perp$ is the cross radius vector, and k_1 and k_N are the wave numbers of the basic and Nth harmonics, respectively.

Let us write the polarization response of the medium as

$$P = P_1^{\text{lin}} + P_N^{\text{lin}} + P^{\text{non}}, \qquad (3)$$

where P_1^{lin} and P_N^{lin} are the linear polarization responses of a basic component and of the Nth harmonics, respectively, and P^{non} is the nonlinear part of the polarization response.

Let us assume that the density of medium is low, i.e., the corresponding refractive index is near the unit in the spectral range, which covers the frequencies ω and $N\omega$. In this case, we can apply to Eq. (1) an approach of the unidirectional propagation along the z-axis.[34] Then, we obtain approximately

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E}{\partial t} = -\frac{2\pi}{c}\frac{\partial P}{\partial t} + \Delta_\perp \int_{-\infty}^{t} E\, dt'. \qquad (4)$$

Note that this approach is well satisfies to the paraxial approximation [see last term on the right-hand side of Eq. (4)].

We will present the linear parts of polarizing responses in the form of the well-known expansion:[35]

$$P_1^{\text{lin}} \approx \chi_\omega \psi_1 e^{i(\omega t - k_1 z)} - i\frac{\partial \chi_\omega}{\partial \omega}\frac{\partial \psi_1}{\partial t}e^{i(\omega t - k_1 z)} - \frac{1}{2}\frac{\partial^2 \chi_\omega}{\partial \omega^2}\frac{\partial^2 \psi_1}{\partial t^2}e^{i(\omega t - k_1 z)} + \text{c.c.}, \qquad (5a)$$

$$P_N^{\text{lin}} \approx \chi_{N\omega}\psi_N e^{i(N\omega t - k_N z)} - i\frac{\partial \chi_{N\omega}}{\partial (N\omega)}\frac{\partial \psi_N}{\partial t}e^{i(N\omega t - k_N z)} - \frac{1}{2}\frac{\partial^2 \chi_{N\omega}}{\partial (N\omega)^2}\frac{\partial^2 \psi_N}{\partial t^2}e^{i(N\omega t - k_N z)} + \text{c.c.} \qquad (5b)$$

Here, χ_ω and $\chi_{N\omega}$ are the linear susceptibilities on the frequencies ω and $N\omega$, respectively.

In turn, neglecting the dependence of the nonlinear susceptibility $\chi^{(N)}$ on the frequency, we will present a nonlinear part of the polarization response in the following simple form:

$$P^{\text{non}} = \chi^{(N)} E^N. \tag{6}$$

The physical mechanisms resulting in quadratic and cubic nonlinearities can have various origins.[2] As for nonlinearities of higher orders ($N \geq 4$), the major role in their emergence is played by the electron-optical intra-atomic mechanism. In anisotropic media, this mechanism can lead to the generation of both even and odd harmonics.

Substituting Eq. (2) into Eq. (6) and selecting only terms that are oscillating on the frequencies ω and $N\omega$, we will have

$$P^{\text{non}} = P_1^{\text{non}} + P_N^{\text{non}}, \tag{7}$$

where

$$P_1^{\text{non}} = \chi^{(N)} \left[N\psi_1^{*N-1}\psi_N e^{i(Nk_1 - k_N)z} + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!|\psi_1|^{N-1-2l}|\psi_N|^{2l}\psi_1}{((N-1)/2 - l)!((N+1)/2 - l)!l!^2} \right] e^{i(\omega t - k_1 z)} + \text{c.c.}, \tag{8}$$

$$P_N^{\text{non}} = \chi^{(N)} \left[\psi_1^N e^{-i(Nk_1 - k_N)z} + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!|\psi_1|^{N-1-2l}|\psi_N|^{2l}\psi_N}{((N-1)/2 - l)!^2 l!(l+1)!} \right] e^{i(N\omega t - k_N z)} + \text{c.c.}, \tag{9}$$

$\varepsilon = 0$, if N is even, and $\varepsilon = 1$, if N is odd.

We will assume further that the conditions of the matching of the phase and group velocities are satisfied, i.e.,

$$k_N = Nk_1, \tag{10}$$

$$v_{g1} = v_{gN}, \tag{11}$$

where the group velocity v_{g1} is defined by the expression

$$\frac{1}{v_{g1}} = \frac{\partial k_1}{\partial \omega} = \frac{1}{c}\left[1 + 2\pi\left(\chi_\omega + \omega\frac{\partial \chi_\omega}{\partial \omega}\right)\right];$$

the expression for the wave number k_1 is $k_1 = (\omega/c)(1 + 2\pi\chi_\omega)$. The formulae for v_{gN} and k_N are derived from these expressions for v_{g1} and k_1 by the replacement $\omega \to N\omega$.

In the substitution of Eqs. (8) and (9) into Eq. (4), we assume that $\partial P_1^{\text{non}}/\partial t \approx i\omega P_1^{\text{non}}$ and $\partial P_N^{\text{non}}/\partial t \approx iN\omega P_N^{\text{non}}$. In linear terms, we will neglect the temporary derivatives more than the second order [see Eqs. (5a) and (5b)]. Then, we obtain the set of two wave equations:

$$i\frac{\partial \psi_1}{\partial z} = -\frac{k_2^{(1)}}{2}\frac{\partial^2 \psi_1}{\partial \tau^2} + b\left[N\psi_1^{*N-1}\psi_N + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!|\psi_1|^{N-1-2l}|\psi_N|^{2l}\psi_1}{((N-1)/2 - l)!((N+1)/2 - l)!l!^2} \right] + \frac{c}{2\omega}\Delta_\perp \psi_1, \tag{12}$$

$$i\frac{\partial \psi_N}{\partial z} = -\frac{k_2^{(N)}}{2}\frac{\partial^2 \psi_N}{\partial \tau^2} + Nb\left[\psi_1^N + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!|\psi_1|^{N-1-2l}|\psi_N|^{2l}\psi_N}{((N-1)/2 - l)!^2 l!(l+1)!} \right] + \frac{c}{2N\omega}\Delta_\perp \psi_N, \tag{13}$$

where $b = (2\pi\omega/c)\chi_N$ and $\tau = t - z/v_{g1}$. The coefficients of dispersion of the group velocity (DGV), $k_2^{(1)}$ and $k_2^{(N)}$, are defined as

$$k_2^{(1)} = \frac{\partial}{\partial \omega}\left(\frac{1}{v_{g1}}\right) \quad \text{and} \quad k_2^{(N)} = \frac{\partial}{\partial N\omega}\left(\frac{1}{v_{gN}}\right).$$

According to the first terms in the square brackets of equations (12) and (13), a condition of phase matching (10) can be valid only for one degree N. Therefore, other degrees of nonlinearity are absent in (12) and (13).

It is very difficult to meet the conditions (10) and (11). For this purpose, the technique of tilted wave fronts can be used, for example.[2]

The further analysis is based on the investigations of soliton-like solutions of Eqs. (12) and (13).

3. Temporal Steady-State Soliton-Like Solutions

Assuming in (12) and (13) $\Delta_\perp\psi_1 = \Delta_\perp\psi_N = 0$, we will find the one-dimensional soliton-like solutions in the following forms:

$$\psi_1(z,\tau) = F_1(\tau)e^{iq_1 z}, \quad \psi_N(z,\tau) = F_N(\tau)e^{iq_N z}, \tag{14}$$

where q_1 and q_N are some constants and $F_1(\tau)$ and $F_N(\tau)$ are the unknown real functions.

Substituting Eq. (14) into Eqs. (12) and (13), we obtain

$$-q_1 F_1 = -\frac{k_2^{(1)}}{2}\ddot{F}_1 + b\left[NF_1^{N-1}F_N e^{i(q_N - Nq_1)z} + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!F_1^{N-2l}F_N^{2l}}{((N-1)/2 - l)!((N+1)/2 - l)!l!^2} \right], \tag{15}$$

$$-q_N F_N = -\frac{k_2^{(N)}}{2}\ddot{F}_N + Nb\left[F_1^N e^{-i(q_N - Nq_1)z} + \varepsilon \sum_{l=0}^{(N-1)/2} \frac{N!F_1^{N-1-2l}F_N^{2l+1}}{((N-1)/2 - l)!^2 l!(l+1)!} \right]. \tag{16}$$

By assuming in Eqs. (15) and (16)

$$q_N = Nq_1, \tag{17}$$

we have $e^{\pm i(q_N - Nq_1)z} = 1$.

We will consider further that the functions $F_1(\tau)$ and $F_N(\tau)$ are proportional to each other, i.e.,

$$F_N = A_N F_1, \quad (N = 1, 2, 3, \ldots), \tag{18}$$

where A_N is the constant, defined below.

Then, the self-consistence of Eqs. (15) and (16) requires the performance of the equalities

$$k_2^{(N)} = N k_2^{(1)}, \tag{19}$$

$$N A_N - \frac{1}{A_N} = \varepsilon(\Sigma_N - \Sigma_1), \tag{20}$$

where

$$\Sigma_1 = \sum_{l=0}^{(N-1)/2} \frac{N! A_N^{2l}}{((N-1)/2 - l)!((N+1)/2 - l)! l!^2},$$

$$\Sigma_N = \sum_{l=0}^{(N-1)/2} \frac{N! A_N^{2l}}{((N-1)/2 - l)! l! (l+1)!}. \tag{21}$$

As a result, for $F_1(\tau)$, we come to the equation

$$\ddot{F}_1 = \frac{1}{\tau_\mathrm{p}^2} F_1 - \alpha F_1^N, \tag{22}$$

where

$$\alpha = -\frac{2b}{k_2^{(1)}}(N A_N + \varepsilon \Sigma_1), \tag{23}$$

and the temporary duration τ_P of a soliton is determined as

$$\frac{1}{\tau_\mathrm{p}^2} = \frac{2q_1}{k_2^{(1)}} = \frac{2q_N}{k_2^{(N)}}. \tag{24}$$

The soliton-like solution of Eq. (22) is

$$F_1 = \left(\frac{N+1}{2\alpha\tau_\mathrm{p}^2}\right)^{\frac{1}{N-1}} \left[\mathrm{sech}\left(\frac{N-1}{2}\frac{\tau}{\tau_\mathrm{p}}\right)\right]^{\frac{2}{N-1}}. \tag{25}$$

From here, and also from Eqs. (14), (17), and (18), we find

$$\psi_1 = \left(\frac{N+1}{2\alpha\tau_\mathrm{p}^2}\right)^{\frac{1}{N-1}} \exp\left(i\frac{k_2^{(1)}}{2}z\right)\left[\mathrm{sech}\left(\frac{N-1}{2\tau_\mathrm{p}}\tau\right)\right]^{\frac{2}{N-1}}, \tag{26}$$

$$\psi_N = A_N \left(\frac{N+1}{2\alpha\tau_\mathrm{p}^2}\right)^{\frac{1}{N-1}} \exp\left(i\frac{N k_2^{(1)}}{2}z\right)\left[\mathrm{sech}\left(\frac{N-1}{2\tau_\mathrm{p}}\tau\right)\right]^{\frac{2}{N-1}}. \tag{27}$$

We will analyze separately the cases of the generation of even and odd harmonics.

3.1 *Even harmonics ($N = 2, 4, 6, \ldots$)*

In this case, $\varepsilon = 0$. Then, from Eqs. (20) and (23), it follows that

$$A_N = \pm\frac{1}{\sqrt{N}}, \quad \alpha = \mp\frac{2b}{\sqrt{N} k_2^{(1)}}. \tag{28}$$

Thus, the efficiency of the generation of the harmonics decreases with the increase in their serial number N. In turn, from Eqs. (26) and (27), it is visible that parameter α can be both positive and negative.

3.2 *Odd harmonics ($N = 3, 5, 7, \ldots$)*

In this case, $\varepsilon = 1$. Then, the condition (20) represents the algebraic equation of the degree N, from which the coefficient A_N is defined. We will postpone the study of the roots of Eq. (20) until the following section. Here, we will only notice that these roots exist for any positive odd values of N.

The expression in brackets of the right-hand side of Eq. (23) for all values of A_N is positive. From Eqs. (25) and (26), it follows that $\alpha > 0$. Therefore, solutions (26) and (27) exist if $b/k_2^{(1)} \sim \chi^{(N)}/k_2^{(1)} < 0$. This condition is well known from the theory of optical solitons: if the nonlinearity is focusing ($\chi^{(N)} > 0$), then solitons are formed in the spectral area of negative DGV ($k_2^{(1)} < 0$) and vice versa. It is seen that this property is fair also for the pulse mode of the generation of odd harmonics.

Let us analyze a possibility of the fulfillment of the conditions (10), (11), and (19). Let the dispersion of the medium be determined by one spectral line of resonant absorption on the characteristic frequency ω_0. Then, Sellmeier's formula becomes

$$\chi_\omega = \frac{\omega_0^2 \chi_0}{\omega_0^2 - \omega^2},$$

where χ_0 is the linear susceptibility of the medium at $\omega = 0$.

Let the basic frequency and the frequency of the harmonics satisfy the condition $\omega \ll \omega_0$. Then we have $\chi_\omega \approx \chi_0(1 + \omega^2/\omega_0^2)$. If the medium is microdispersive (granulated), then a spatial dispersion is essential. Then, the last formula can be modified as[36]

$$\chi_\omega \approx \chi_0\left(1 + \eta\frac{\omega^2}{\omega_0^2}\right), \tag{29}$$

where η is the dimensionless constant, the absolute value of which is a unit order; in the presence of spatial dispersion, the parameter η can be both positive and negative.[36]

In this case, we have

$$k_1 = \frac{\omega}{c}\left[1 + 2\pi\chi_0\left(1 + \eta\frac{\omega^2}{\omega_0^2}\right)\right],$$

$$\frac{1}{v_{\mathrm{g}1}} = \frac{1}{c}\left[1 + 2\pi\chi_0\left(1 + 3\eta\frac{\omega^2}{\omega_0^2}\right)\right],$$

$$k_2^{(1)} = \frac{12\pi\chi_0\eta}{c\omega_0^2}\omega. \tag{30}$$

Because $k_2^{(1)}$ is proportional to the carrier frequency ω, condition (19) is satisfied automatically. Note that the parameters $k_2^{(1)}$ and $k_2^{(N)}$ can be both positive and negative. At the same time, it is not possible to meet precisely conditions (10) and (11). However, taking into account that $2\pi\chi_0 \ll 1$ and $\omega^2/\omega_0^2 \ll 1$, it is possible to consider that conditions (10) and (11) are satisfied approximately. We have then $k_1 \approx n\omega/c$, $k_N \approx N k_1$, and $v_{\mathrm{g}1} \approx v_{\mathrm{g}N} \approx c/n$, where $n = 1 + 2\pi\chi_0$ is the inertia-less part of the refractive index. Let us reveal the conditions under which the small detuning of phase and group velocities can be neglected. Let l_s be the medium sample scale length in the direction of pulse propagation. Then, $t_1 = l_\mathrm{s}/v_{\mathrm{g}1}$ and $t_N = l_\mathrm{s}/v_{\mathrm{g}N}$ are the propagation times through the medium of the basic pulse and pulse of harmonics, respectively. The detuning of velocities is not significant if $|t_1 - t_N| \ll \tau_\mathrm{p}$. From here and from (30), we find

$$l_\mathrm{s} \ll \left(\frac{\omega_0}{\omega}\right)^2 \frac{c\tau_\mathrm{p}}{6\pi|\eta|\chi_0(N^2 - 1)}.$$

The accounting for phase detuning leads to a similar inequality. Assuming that $\omega_0/\omega \sim 10$–10^2, $N \sim 10$, $\chi_0 \sim 10^{-2}$, and $\tau_p \sim 100$ fs, we will obtain $l_s \ll 1$–10 cm.

4. Spatiotemporal Solitons

We investigate the effect of spatial perturbations on the temporary solitons (26) and (27) now, assuming in Eqs. (12) and (13) that $\Delta_\perp\psi_{1,N} \neq 0$. For this purpose, we use a method of the averaged Lagrangian.[37,38] Note in the beginning that Eqs. (12) and (13) correspond to the Lagrangian density

$$L = L_1 + L_N + L_{\rm int}, \tag{31}$$

where

$$L_1 = \frac{i}{2}\left(\psi_1^*\frac{\partial\psi_1}{\partial z} - \psi_1\frac{\partial\psi_1^*}{\partial z}\right) - \frac{k_2^{(1)}}{2}\left|\frac{\partial\psi_1}{\partial\tau}\right|^2 + \frac{c}{2\omega}|\nabla_\perp\psi_1|^2, \tag{32}$$

$$L_N = \frac{i}{2N}\left(\psi_N^*\frac{\partial\psi_N}{\partial z} - \psi_N\frac{\partial\psi_N^*}{\partial z}\right) - \frac{k_2^{(N)}}{2N}\left|\frac{\partial\psi_N}{\partial\tau}\right|^2 + \frac{c}{2N^2\omega}|\nabla_\perp\psi_N|^2, \tag{33}$$

$$L_{\rm int} = -b(\psi_1^{*N}\psi_N + \psi_1^N\psi_N^*) - b\sum_{l=0}^{(N+1)/2}\frac{N!|\psi_1|^{N+1-2l}|\psi_N|^{2l}}{((N+1)/2-l)!^2 l!^2}. \tag{34}$$

We will write trial solutions, taking into account Eqs. (26) and (27). Then

$$\psi_1 = \left(\frac{N+1}{2\alpha}\right)^{\frac{1}{N-1}} Q^{\frac{2}{N-1}} e^{i\varphi}\left[\operatorname{sech}\left(\frac{N-1}{2}Q\tau\right)\right]^{\frac{2}{N-1}}, \tag{35}$$

$$\psi_N = A\left(\frac{N+1}{2\alpha}\right)^{\frac{1}{N-1}} Q^{\frac{2}{N-1}} e^{iN\varphi}\left[\operatorname{sech}\left(\frac{N-1}{2}Q\tau\right)\right]^{\frac{2}{N-1}}, \tag{36}$$

where Q and φ are the unknown functions of coordinates.

In a one-dimensional case ($\Delta_\perp\psi_{1,N} = 0$), we have $Q = 1/\tau_p$, $\varphi = k_2^{(1)} z/(2\tau_p^2)$. Therefore, we will call Q and φ the "slow" and "fast" functions of coordinates, respectively.[37]

According to the method of the averaged Lagrangian, we will substitute Eqs. (35) and (36) into Eqs. (31)–(34). After this, we will integrate the obtained expression on τ. As a result, we obtain

$$\int_{-\infty}^{+\infty} L d\tau = 2^{\frac{4}{N-1}}\frac{1+A_N^2}{N-1}\left(\frac{N+1}{2\alpha}\right)^{\frac{2}{N-1}}\frac{\Gamma^2\left(\dfrac{2}{N-1}\right)}{\Gamma\left(\dfrac{4}{N-1}\right)}\Lambda,$$

where $\Gamma(\zeta)$ is Euler's gamma function, and Λ is the averaged Lagrangian, which can be written down in the form of the sum of the refraction Λ_R and the diffraction Λ_D:

$$\Lambda = \Lambda_R + \Lambda_D, \tag{37}$$

$$\Lambda_R = -Q^{\frac{5-N}{N-1}}\frac{\partial\varphi}{\partial z} + \frac{c}{2\omega}Q^{\frac{5-N}{N-1}}(\nabla_\perp\varphi)^2 + \frac{k_2^{(1)}}{2}\sigma_N Q^{\frac{N+3}{N-1}}, \tag{38}$$

$$\Lambda_D = \frac{c}{4\omega}\frac{1+A_N^2/N^2}{1+A_N^2}\left(\frac{2}{N-1}\right)^2(3-N+B_N)Q^{\frac{7-3N}{N-1}}(\nabla_\perp Q)^2. \tag{39}$$

Here,

$$\sigma_N = \frac{1}{(N+3)}\left(1-N+2\frac{N+1}{1+A_N^2}\frac{2A_N+\varepsilon\Sigma}{NA_N+\varepsilon\Sigma_1}\right),\quad \Sigma = \sum_{l=0}^{(N+1)/2}\frac{N!A_N^{2l}}{((N+1)/2-l)!^2 l!^2},$$

$$B_N = \frac{(N-1)^3}{4}\frac{\Gamma(4/(N-1))}{\Gamma^2(2/(N-1))}\left[{}_4F_3\left(\frac{2}{N-1},\frac{2}{N-1},\frac{2}{N-1},\frac{4}{N-1};\frac{N+1}{N-1},\frac{N+1}{N-1},\frac{N+1}{N-1};-1\right)\right.$$
$$\left. - \frac{32}{(N+1)^3}\,{}_4F_3\left(\frac{N+1}{N-1},\frac{N+1}{N-1},\frac{N+1}{N-1},2\frac{N+1}{N-1};\frac{2N}{N-1},\frac{2N}{N-1},\frac{2N}{N-1};-1\right)\right],$$

${}_4F_3(\zeta_1,\zeta_2,\zeta_3,\zeta_4;\xi_1,\xi_2,\xi_3;-1)$ is the generalized hypergeometrical function.[39]

For obtaining Eqs. (37)–(39), we used the definite integrals given in Appendix.

Let us simplify the expression for the coefficient σ_N. Let the value N be an even number. Then, assuming that $\varepsilon = 0$ and taking into account expression (28), we will have

$$\sigma_N = \frac{5-N}{N+3}. \tag{40}$$

Let N now be an odd number. It is easy to see that

$$\Sigma_1 = \frac{N+1}{2}\Sigma - A_N^2\Sigma_N.$$

From here and from (20), we find

$$\Sigma_1 = \frac{1}{1+A_N^2}\left(\frac{N+1}{2}\Sigma + A_N - NA_N^3\right).$$

Then, the expression for σ_N coincides with the Eq. (40) again.

Having made the replacements

$$\rho = Q^{\frac{5-N}{N-1}},\quad \Phi = -\frac{c}{\omega}\varphi, \tag{41}$$

we will rewrite the averaged Lagrangian as

$$\Lambda = \rho\frac{\partial\Phi}{\partial z} + \rho\frac{(\nabla_\perp\Phi)^2}{2} + \frac{ck_2}{2\omega}\frac{5-N}{N+3}\rho^{\frac{N+3}{5-N}} + G\frac{(\nabla_\perp\rho)^2}{4\rho}, \tag{42}$$

where

$$G = \left(\frac{2c}{\omega}\right)^2\frac{1+A_N^2/N^2}{(1+A_N^2)(5-N)^2}(3-N+B_N).$$

Using Eq. (42), we will write down the system of the equations of Euler-Lagrange for ρ and Φ:

$$\frac{\partial}{\partial z}\frac{\partial\Lambda}{\partial(\partial\Phi/\partial z)} + \nabla_\perp\frac{\partial\Lambda}{\partial(\nabla_\perp\Phi)} = 0,\quad \frac{\partial\Lambda}{\partial\rho} - \nabla_\perp\frac{\partial\Lambda}{\partial(\nabla_\perp\rho)} = 0.$$

Using the equality

$$\frac{\Delta_\perp\rho}{\rho} - \frac{(\nabla_\perp\rho)^2}{2\rho} = 2\frac{\Delta_\perp\sqrt{\rho}}{\sqrt{\rho}},$$

we will have as a result

$$\frac{\partial\rho}{\partial z} + \nabla_\perp(\rho\nabla_\perp\Phi) = 0, \tag{43}$$

$$\frac{\partial\Phi}{\partial z} + \frac{(\nabla_\perp\Phi)^2}{2} + \frac{ck_2^{(1)}}{2\omega}\rho^{2\frac{N-1}{5-N}} = G\frac{\Delta_\perp\sqrt{\rho}}{\sqrt{\rho}}. \tag{44}$$

In a one-dimensional case ($\nabla_\perp \equiv 0$) from (43), (44), and (41), we find $Q = 1/\tau_p = \text{const.}$, $\varphi = k_2^{(1)}Q^2 z/2 = k_2^{(1)}z/(2\tau_p^2)$, which coincides with the parameters of the exact one-dimensional solutions (26) and (27). This circumstance is an important argument in favor of the correctness of the averaged Lagrangian method.

If $G = 0$, the system of Eqs. (43) and (44) formally describes a nonstationary two-dimensional current of the ideal liquid, where the role of time is played by the z-coordinate. The first equation is the continuity equation; the second equation is Cauchy's integral. If $G \neq 0$, Eqs. (43) and (44) are similar to the equations of a current of quantum liquid with the internal interactions.[40] We will notice that the case of $G = 0$ corresponds to the approximation of "geometrical optics".[41] In turn, the right-hand side of (44) considers the effects of solitonic diffraction.

4.1 Approximation of "geometrical optics"

In the beginning, we will find the axially symmetrical solutions of the Eqs. (41), (43), and (44) in the approximation of nonlinear refraction, neglecting diffraction. It is clear that only localized solutions for Q have a physical meaning. These solutions have to possess limited energy. We will find such solutions in Ref. 42

$$\rho = \rho_0\frac{R_0^2}{R^2(z)}F\left(\frac{r}{R(z)}\right), \quad \Phi = f(z) + \frac{r^2}{2R}\frac{dR}{dz}, \tag{45}$$

where $\rho_0 = \text{const.}$, r is the radial component of the cylindrical system of coordinates, and R, F, and f are the still unknown functions; R has the meaning of a cross radius (aperture) of a soliton, and R_0 is the initial soliton aperture.

The solution (45) identically satisfies Eq. (43). From Eqs. (35), (36), and the second expression (45), it is seen that the dynamic parameter $(1/R)(dR/dz)$ has the meaning of the curvature of the soliton wave fronts. In turn, df/dz is a nonlinear correction to the refractive index.

Substituting Eq. (45) into Eq. (44) and assuming that $G = 0$, we obtain

$$\frac{df}{dz} + \frac{r^2}{2R}\frac{d^2R}{dz^2} + \frac{ck_2^{(1)}}{2\omega}\left(\rho_0\frac{R_0^2}{R^2}\right)^{2\frac{N-1}{5-N}}F^{2\frac{N-1}{5-N}} = 0. \tag{46}$$

We will choose F as

$$F = \left(1 - \frac{r^2}{R^2}\right)^{\frac{5-N}{2(N-1)}}. \tag{47}$$

Then from Eq. (46) we obtain

$$\frac{df}{dz} = -\frac{ck_2^{(1)}}{2\omega}Q_0^2\left(\frac{R_0}{R}\right)^{4\frac{N-1}{5-N}}, \tag{48}$$

$$\frac{d^2R}{dz^2} = -\frac{\partial U_R}{\partial R}. \tag{49}$$

Here, $Q_0 = \rho_0^{\frac{N-1}{5-N}}$,

$$U_R = \frac{ck_2^{(1)}}{4\omega}\frac{5-N}{N-1}Q_0^2\left(\frac{R_0}{R}\right)^{4\frac{N-1}{5-N}}. \tag{50}$$

At the same time, according to the first expression (41), we have

$$Q = Q_0\left(\frac{R_0}{R}\right)^{2\frac{N-1}{5-N}}\sqrt{1 - \frac{r^2}{R^2}}, \quad r \leq R. \tag{51}$$

It is obvious that Q_0^{-1} is the initial temporary duration of a soliton on the axis $r = 0$. At the same time, the parameter $Q_0^{\frac{2}{N-1}}$ is proportional to the initial amplitudes of both soliton components at $r = 0$. In turn, the dynamic parameters Q^{-1} and $Q^{\frac{2}{N-1}}$ are the local duration and amplitude, respectively [see Eqs. (35) and (36)].

Equation (49) is formally similar to the motion equation of the Newtonian particle of a unit mass in the field with the potential energy U_R. Using this analogy, we will analyze on the basis of expressions (50) and (51) the qualitative behavior of the spatial-temporal dynamics of the soliton (35) and (36) under various degrees of the nonlinearity N and under various signs of the parameter $k_2^{(1)}$.

1) $N < 5$, $k_2^{(1)} > 0$. In this case, it follows from expression (50) that the propagation of a soliton is accompanied by an increase in its aperture R. This corresponds to the mode of cross defocusing. At the same time, apparently from Eq. (51) the parameter Q is decreasing. Thus, the temporal duration of a soliton increases, and its amplitude decreases.

2) $N < 5$, $k_2^{(1)} < 0$. Here, the propagation is observed in the self-focusing mode. This mode is followed by the longitudinal compression of a soliton and its peak amplification (transversal–longitudinal collapse).

3) $N > 5$, $k_2^{(1)} > 0$. Here, we have the transversal defocusing, which is accompanied by the longitudinal collapse. Apparently from Eq. (51), this process is followed by an increase in the parameter Q. It corresponds to the longitudinal compression and the peak amplification of the soliton. Thus, the soliton is focused not in a point, and is going to the line, which is perpendicular to the direction of propagation.

4) $N > 5$, $k_2^{(1)} < 0$. In this case, we have the self-focusing mode with respect to transversal directions, which is followed by the longitudinal broadening of a soliton (transversal collapse). Here, the soliton is going to the line parallel to the direction of propagation.

4.2 Diffraction effect

Diffraction can counteract the process of self-focusing (collapse). We investigate this question in detail. Let the right-hand side of Eq. (44) be not equal to zero, i.e., $G \neq 0$. Following the logic of previous works[41–43] and starting from expression (47), we will consider that

$$F = \exp\left[-\frac{5-N}{2(N-1)}\frac{r^2}{R^2}\right]. \tag{52}$$

Then,

$$Q = Q_0\left(\frac{R_0}{R}\right)^{2\frac{N-1}{5-N}}\exp\left(-\frac{r^2}{2R^2}\right). \tag{53}$$

We will particularly note that the expressions (45) and (52) satisfy in accuracy Eq. (43). Moreover, it is seen that the expressions (47) and (52) become similar to each other under the near-axis condition:[42]

$$(r/R)^2 \ll 1. \tag{54}$$

This remark also concerns the expressions (51) and (53).

Table I. Dependences of the coefficients A_N and g_N on even numbers of the generated harmonics.

N	A_N	g_N
2	±0.71	1.00
4	±0.50	0.14
6	±0.41	0.13
8	±0.35	0.15
10	±0.32	0.16
12	±0.29	0.17
∞	0	0.25

We will substitute Eqs. (45) and (52) into Eq. (44). Then, we will expand F into a series with respect to $(r/R)^2$, being limited to the first degree of this parameter [see (54)]. Equating the coefficients at r^0 and r^2 in the left- and right-hand sides, we will obtain

$$\frac{df}{dz} = -\frac{ck_2^{(1)}}{2\omega}Q_0^2\left(\frac{R_0}{R}\right)^{4\frac{N-1}{5-N}} - \left(\frac{c}{\omega}\right)^2\frac{h_N}{R^2}, \tag{55}$$

where

$$h_N = 4\,\frac{1 + A_N^2/N^2}{1 + A_N^2}\,\frac{3 - N + B_N}{(5-N)(N-1)}.$$

For the aperture, we obtain equation (49), taking into account the replacement $U_{\rm R} \to U$, where

$$U = U_{\rm R} + U_{\rm D}, \tag{56}$$

$$U_{\rm D} = \left(\frac{c}{\omega}\right)^2\frac{g_N}{R^2}, \tag{57}$$

$$g_N = \frac{(1 + A_N^2/N^2)(3 - N + B_N)}{(1 + A_N^2)(N-1)^2},$$

and $U_{\rm R}$ is determined by the Eq. (50).

The part $U_{\rm D}$ of potential energy U is accounted for the effect of diffraction.

The first integral of equation (49) with replacing $U_{\rm R} \to U$ has the appearance

$$\frac{1}{2}\left(\frac{dR}{dz}\right)^2 + U(R) = \frac{1}{2}\left(\frac{dR}{dz}\right)_{z=0} + U(R_0). \tag{58}$$

The simple analysis shows that the coefficient g_N is positive for all values of N. It concerns both even and odd harmonics. The values of the coefficients g_N and A_N are given in Tables I and II for even and odd values of N, respectively. The coefficient A_N in the case of even values of N is determined using Eq. (28). In case of odd values of N, this coefficient is a root of equation (20), where $\varepsilon = 1$.

In the case of even harmonics, the diffraction coefficient g_N has the maximum value $g_N = 1.00$ at $N = 2$. Upon reaching the minimum at $N = 6$, this coefficient then monotonically increases, aspiring in a limit $N \to \infty$ to the value of 0.25. In turn, the coefficient A_N monotonically decreases with the growth of the number of a harmonic according to Eq. (28). The sign "±" on the right-hand side of Eq. (28) reflects the fact that under condition (17), the set of Eqs. (15) and (16) is invariant with respect to the transformations $F_1 \to -F_1$ and $F_N \to F_N$ at the even values of N. The efficiency of the generation of the harmonics is defined by the ratio of the intensities $\sim(F_N/F_1)^2$. Therefore, the choice of a sign in (28) is not important.

Table II. Dependences of the coefficients A_N and g_N on odd numbers of the generated harmonics.

N	A_N	g_N
3	−1.52	0.16
	−0.28	0.28
	0.79	0.20
5	−1.05	0.08
	−0.05	0.15
	0.95	0.08
7	−1.01	0.08
	−0.01	0.16
	0.99	0.08
9	±1	0.09
	0	0.17
∞	±1	0.125
	0	0.25

In the case of odd harmonics to each value of N, there correspond three values of the coefficient A_N: $A_N^{(1)}$, $A_N^{(2)}$, and $A_N^{(3)}$. It is also seen that $A_N^{(1)} \to -1$, $A_N^{(2)} \to 0$, and $A_N^{(3)} \to 1$ with the increase in the value of N. To understand what value of A_N should be chosen under each fixed value of N, we will address formulae (20) and (21). In Figs. 1–5, the dependences of the potential energy $U(R)$ corresponding to these formulae are represented. Under the condition $k_2^{(1)} > 0$, the function $U(R)$ monotonically decreases with the increase in the aperture R for all values of N (Figs. 1 and 2). It corresponds to the unlimited broadening of the pulse.

Thus, under the condition $k_2^{(1)} > 0$, the formation of stable spatiotemporal solitons is impossible. On the other hand, the function $U(R)$ under the condition $k_2^{(1)} < 0$ possesses a local minimum for all even and odd values of N, except $N = 3$ and $N = 4$ (see Figs. 3–5).

The case of $N = 5$ in our approach is special. Here, additional investigation by more exact methods is required.

The existence of a local minimum in the function $U(R)$ corresponds to a possibility of the formation of a stable spatiotemporal soliton or a light bullet (Figs. 3 and 5). The existence of a local maximum in the function $U(R)$ attests to the instability of the pulse mode of the generation of the harmonics for $N = 3$ and $N = 4$ (Fig. 4). As noted in Introduction, under the generation of third harmonics, the observed light bullet changed its shape nonperiodically.[31] This observation coincides with the result obtained here: in the course of the generation of the third harmonics, the steady-state spatiotemporal solitons cannot be formed.

In the case of odd values of N, the stable pulse mode is possible under the condition $N \geq 7$. From Table I, it is seen that, in view of the approximate character of the approaches used here, it is possible with good accuracy for the odd values of N to assume that $A_N = \pm 1$. Here, we have also considered the remark on the insignificance of a sign of the coefficient A_N made above. At the same time, the values of the coefficient g_N for various numbers N of odd harmonics lie in an interval between 0.08 and 0.125 (see Table II). The case $A_N = 0$ corresponds to the lack of the generation of the harmonics. Therefore, this case is out of consideration. The pulse mode of generation for $N = 3$ is unstable.

We will find the value $R_m^{(N)}$ corresponding to the extremum of $U(R)$ from the condition $\partial U/\partial R = 0$. By assuming also that $R_0 = R_m^{(N)}$, we will have

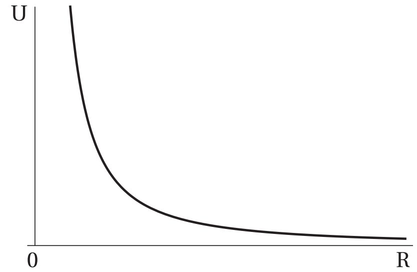

Fig. 1. Schematic dependence of the potential energy U on the pulse aperture R for $2 \le N \le 4$ in the case of $k_2^{(1)} > 0$.

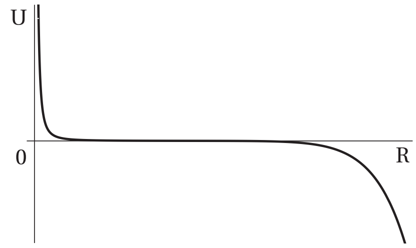

Fig. 2. Schematic dependence of the potential energy U on the pulse aperture R for $N \ge 6$ in the case of $k_2^{(1)} > 0$.

$$R_m^{(N)} = \tau_\mathrm{p}\sqrt{\frac{2cg_N}{\omega|k_2^{(1)}|}}, \tag{59}$$

where $\tau_\mathrm{p} = 1/Q_0$ is the initial temporal duration of a soliton on the central axis (at $r = 0$).

Note that for $N = 2, 6, 7, 8, \ldots$ this value of $R_m^{(N)}$ corresponds to the minimum of the potential energy $U(R)$, and for $N = 3$ and $N = 4$, to the maximum.

In these cases ($N = 3, 4$), under the conditions $R_0 < R_m^{(3,4)}$ $[(\partial U/\partial R)_{R=R_0} < 0]$ and $k_2^{(1)} < 0$, both pulse components experience irreversible spatiotemporal broadening. Otherwise, i.e., if $R_0 > R_m^{(3,4)}$ $(\partial U/\partial R)_{R=R_0} > 0$, we have a collapse of the laser pulse. Using (23), (26), (27), and the expression for b [see below equation (13)], and taking into account (59), we will rewrite the conditions of the appearance of a collapse as

$$\pi|\psi_1|^2 R_m^{(3)2} \sim \frac{c^2 g_3}{\omega^2\chi_3}, \quad \pi|\psi_1|^3 R_m^{(4)2} \sim \frac{c^2 g_4}{\omega^2\chi_4}.$$

Introducing the intensity $I_1 = (c/4\pi)|\psi_1|^2$, in the case of cubic nonlinearity, we will have from here the known condition on power[2] $P_1 \sim \pi I_1 R_0^2 > \pi I_1 R_m^{(3)2} \sim P_c = c\lambda^2/(48\pi^3\chi_3)$, where $\lambda = 2\pi c/\omega$ is the wavelength of the basic pulse component. This coincidence serves as an argument in favor of the results obtained here.

Using the last Eq. (30) here, we will rewrite Eq. (59) as

$$R_m^{(N)} = c\tau_\mathrm{p}\frac{\omega_0}{\omega}\sqrt{\frac{g_N}{6\pi\chi_0|\eta|}}. \tag{59a}$$

Within our physical model, $\omega_0/\omega \ll 1$ and $g_N/(6\pi\chi_0|\eta|) \sim 1$ (see Sect. 3). Because the longitudinal scale length of a soliton $l_\parallel \sim c\tau_\mathrm{p}$, from Eq. (59a), it follows that $R_m^{(N)} \ll l_\parallel$. By assuming also that $\tau_\mathrm{p} \sim 100\,\mathrm{fs}$ and $\omega_0/\omega \sim 10$–10^2, we will have $R_m^{(N)} \sim 0.1$–1 mm.

The case of $N = 2$ under the condition $k_2^{(1)} < 0$ has one important difference from the cases when $N = 6, 7, 8, \ldots$.

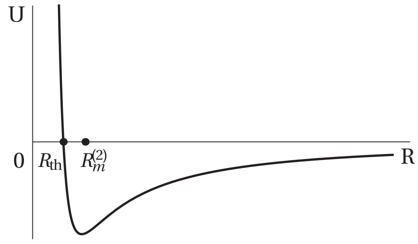

Fig. 3. Dependence of the potential energy U on the pulse aperture R for $N = 2$ in the case of $k_2^{(1)} < 0$.

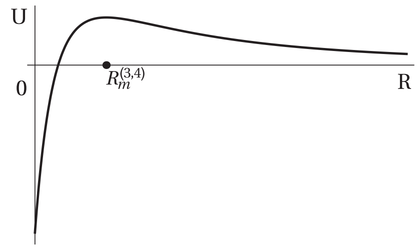

Fig. 4. Schematic dependence of the potential energy U on the pulse aperture R for $N = 3$ and $N = 4$ in the case of $k_2^{(1)} < 0$.

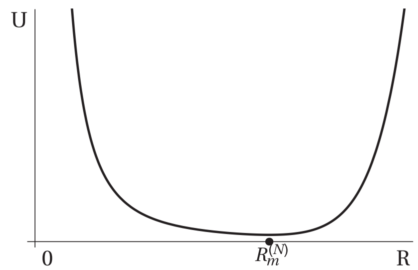

Fig. 5. Schematic dependence of the potential energy U on the pulse aperture R for $N \ge 6$ in the case of $k_2^{(1)} < 0$.

Actually, in the case of $N = 2$, the potential energy $U(R) \to 0$, if $R \to \infty$. For $N = 6, 7, 8, \ldots$ in a limit $R \to \infty$, we have $U(R) \to \infty$. Therefore, in the case of $N = 6, 7, 8, \ldots$ under the condition $k_2^{(1)} < 0$, the light bullet can be created at any input values of R_0 and $(dR/dz)_{z=0}$. This statement is incorrect for the case of $N = 2$. Let, for example, the wave fronts of the input pulse be flat. This means that $(dR/dz)_{z=0} = 0$ [see the second expression (45)]. Then, the stable bullet can be created when performing the condition $R_0 > R_\mathrm{th} = \sqrt{2/3}R_m^{(2)} \approx 0.82R_m^{(2)}$. Here, the value of R_th is determined from the condition $U(R_\mathrm{th}) = 0$ at $R = R_0$. Otherwise, the pulse broadens in all directions.

If $R_0 \ne R_m^{(N)}$ and $N \ne 3, 4, 5$, then the aperture of the "light bullet" under propagation in the medium oscillates around the value of $R_m^{(N)}$. The oscillations of aperture will be accompanied by the oscillations of the duration $1/Q$ of a soliton, its amplitude $Q^{\frac{2}{N-1}}$, and also periodic changes in its phase velocity and the curvatures of wave fronts [see Eqs. (53), (35), (36), (45), and (55)].

In the case of $N = 2$, the transversal compression of a light bullet is accompanied by the longitudinal compression and by the peak amplification of both soliton components, and vice versa. If $N = 6, 7, 8, \ldots$, the transversal compression is accompanied by the longitudinal broadening and by the peak decrease of both bullet components. In turn, the transversal broadening is accompanied by the longitudinal compression and by peak amplification [see Eqs. (53), (35), and (36)]. These phenomena occur in the periodic regimes of pulsations of the light bullets.

5. Conclusions

In this work, the soliton-like modes of the generation of the highest harmonics are investigated. Within the offered model, there is only one degree of nonlinearity. The value of this degree coincides with the value of the number of generated harmonic. Such a model is the most physically correct for cases when $N = 2$ and $N = 3$. In the case of the direct (not of cascade) generation of harmonics of higher orders ($N = 4, 5, 6$, etc.) besides the nonlinearity of the degree N, it is necessary to consider also the degrees of lower orders. For example, in the investigation of the generation of the fourth harmonic, the nonlinearities of the second, third, and fourth orders should be considered. In our model, no such account is carried out. However, the conducted investigations show that in the case of harmonics with the values of $N \geq 6$, there is a clear tendency to form light bullets under the condition $k_2^{(1)} < 0$. For $N = 3$ within our model, the bullets cannot be formed. At the same time, it is known that light bullets are formed, for example, in the presence of the saturating nonlinearity,[2] but without the generation of the third harmonic. There is hope that in the pulse mode of the generation of the third harmonic, the saturating nonlinearity can also promote the formation of stable spatiotemporal solitons. The solution of this task is of additional interest.

The applicability of this model may be difficult for large values of N. In this case, the basic frequency and its harmonics may belong to the spectral ranges where the linear and nonlinear properties of the medium change radically. It is therefore important to seek ways to study where the conditions (10), (11), and (19) cannot be satisfied simultaneously.

Within the proposed model, it is possible to account for the dispersion of nonlinear susceptibilities. It is easy to see that in the cases of the even values of N, it is necessary to correct Eq. (28):

$$A_N = \pm\sqrt{\frac{\chi^{(N)}(\omega)}{N\chi^{(N)}(N\omega)}}.$$

Here, $\chi^{(N)}(\omega)$ and $\chi^{(N)}(N\omega)$ are the nonlinear susceptibilities for the basic frequency ω and for the harmonic $N\omega$, respectively. For the even values of N the coefficients in Eq. (19) will depend on the frequency. Then, the values of A_N will also depend on frequency. It will strongly complicate the investigation and it cannot be carried out in a general manner.

Equation (56) corresponds to the aberrationless approximation.[1,42] It is also of interest to obtain and analyze the solutions of Eqs. (12) and (13), similarly to localized optical vortices.[2,44,45]

Ansatz (18) is limiting the number of possible solutions of Eqs. (15) and (16). Other solutions of this set, which are beyond *ansatz* (18), are possible. In the choice of *ansatz*, we made a start from the solutions, which were obtained in Ref. 46 for the generation of the second harmonic in the case of light beams.

On the other hand, the investigation conducted here within the proposed model allows us to use the general approach for studying the soliton-like mode of the generation of the highest harmonics.

The main result of the present work is that, within the proposed model, the pulse process of the generation of the highest harmonics can develop in the formation mode of light bullets. First, for this purpose, the dispersion of group velocity both for the main frequency and for the frequency of harmonic has to be negative. Secondly, the formation of stable bullets is possible under the generation of the second harmonics ($N = 2$), and harmonics with values of N satisfying the condition $N \geq 6$. In the cases of $N = 3$ and $N = 4$, the generation is accompanied by the self-focusing or defocusing of the pulses.

The case of $N = 5$ requires a separate investigation. The method of an averaged Lagrangian used here is unsuitable for $N = 5$.

Further study of the formation of the light bullets in the course of the generation of the harmonics, when, on input into a medium, the pulse of the harmonics is absent, is of interest. Most likely, it is possible to conduct such study only by numerical simulations.

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Appendix

The values of definite integrals at $\mu > 0$[39] are

$$\int_{-\infty}^{+\infty} \mathrm{sech}^{\mu}\,\xi d\xi = 2^{\mu-1}\frac{\Gamma^2(\mu/2)}{\Gamma(\mu)},$$

$$\int_{-\infty}^{+\infty} \xi \tanh\xi\, \mathrm{sech}^{\mu}\,\xi\, d\xi = \frac{2^{\mu-1}}{\mu}\frac{\Gamma^2(\mu/2)}{\Gamma(\mu)},$$

$$\int_{-\infty}^{+\infty} \xi^2 \tanh^2\xi\, \mathrm{sech}^{\mu}\,\xi\, d\xi = 2^{\mu-1}\left[\left(\frac{2}{\mu}\right)^3 {}_4F_3(\mu/2, \mu/2, \mu/2, \mu; \mu+1, \mu+1, \mu+1; -1) - \frac{1}{2}\left(\frac{2}{\mu+1}\right)^3 {}_4F_3(\mu+1, \mu+1, \mu+1, 2(\mu+1); \mu+2, \mu+2, \mu+2; -1)\right].$$

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1) Ya. Silberberg, Opt. Lett. **15**, 1282 (1990).
2) Yu. S. Kivshar and G. P. Agraval, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, New York, 2003).
3) D. Mihalache, Rom. J. Phys. **57**, 352 (2012).
4) A. Chong, W. H. Renninger, D. N. Christodoulides, and F. W. Wise, Nat. Photonics **4**, 103 (2010).
5) G. Fibich and B. Ilan, Opt. Lett. **29**, 887 (2004).
6) W. Huang, Adv. Stud. Theor. Phys. **5**, 371 (2011).
7) B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, J. Opt. B **7**, R53 (2005).
8) P. M. Goorjian and Ya. Silberberg, J. Opt. Soc. Am. **14**, 3253 (1997).
9) T. Povich and J. Xin, J. Nonlinear Sci. **15**, 11 (2005).
10) D. Mihalache, D. Mazilu, L.-C. Crasovan, I. Towers, A. V. Buryak, B. A. Malomed, and L. Torner, Phys. Rev. Lett. **88**, 073902 (2002).
11) M. Blaauboer, B. A. Malomed, and G. Kurizki, Phys. Rev. Lett. **84**, 1906 (2000).
12) H. C. Gurgov and O. Cohen, Opt. Express **17**, 7052 (2009).
13) F. Wise and P. Di Trapani, Opt. Photonics News **13** [2], 28 (2002).
14) N. Akhmediev and J. M. Soto-Crespo, Phys. Rev. A **47**, 1358 (1993).
15) D. Mihalache, D. Mazilu, F. Lederer, B. A. Malomed, Y. V. Kartashov, L.-C. Crasovan, and L. Torner, Phys. Rev. E **73**, 025601(R) (2006).
16) L. Torner, S. Carrasco, J. P. Torres, L. C. Crasovan, and D. Mihalache, Opt. Commun. **199**, 277 (2001).
17) A. B. Aceves, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Opt. Lett. **19**, 329 (1994).
18) A. B. Aceves, C. De Angelis, and S. Wabnitz, Opt. Lett. **17**, 1758 (1992).
19) J. X. Xin, Physica D **135**, 345 (2000).
20) I. Gražulevičiūtė, G. Tamošauskas, V. Jukna, A. Couairon, D. Faccio, and A. Dubietis, Opt. Express **22**, 30613 (2014).
21) R. Šuminas, G. Tamošauskas, G. Valiulis, and A. Dubietis, Opt. Lett. **41**, 2097 (2016).
22) A. A. Kanashov and A. M. Rubenchik, Physica D **4**, 122 (1981).
23) K. Hayata and M. Koshiba, Phys. Rev. Lett. **71**, 3275 (1993).
24) L. Bergé, V. K. Mezentsev, J. J. Rasmussen, and J. Wuller, Phys. Rev. A **52**, R28(R) (1995).
25) L. Torner, C. R. Menyuk, W. E. Torruellas, and G. I. Stegeman, Opt. Lett. **20**, 13 (1995).
26) D. Mihalache, D. Mazilu, B. A. Malomed, and L. Torner, Opt. Commun. **159**, 129 (1999).
27) D. V. Skryabin and W. J. Firth, Phys. Rev. Lett. **81**, 3379 (1998).
28) X. Liu, K. Beckwitt, and F. Wise, Phys. Rev. E **62**, 1328 (2000).
29) I. N. Towers, B. A. Malomed, and F. W. Wise, Phys. Rev. Lett. **90**, 123902 (2003).
30) H. Sakaguchi and B. A. Malomed, J. Opt. Soc. Am. B **29**, 2741 (2012).
31) I. Gražulevičiūtė, R. Šuminas, G. Tamošauskas, A. Couairon, and A. Dubietis, Opt. Lett. **40**, 3719 (2015).
32) Z. Xu, Ya. V. Kartashov, L. C. Crasovan, D. Mihalache, and L. Torner, Phys. Rev. E **70**, 066618 (2004).
33) A. N. Bugai and S. V. Sazonov, JETP Lett. **98**, 638 (2014).
34) J. C. Eilbeck, J. D. Gibbon, P. J. Caudrey, and R. K. Bullough, J. Phys. A **6**, 1337 (1973).
35) L. F. Mollenauer and J. P. Gordon, *Solitons in Optical Fibers: Fundamentals and Applications* (Academic Press, New York, 2006).
36) S. V. Sazonov, Opt. Spectrosc. **79**, 260 (1995).
37) S. K. Zhdanov and B. A. Trubnikov, Sov. Phys. JETP **66**, 904 (1987).
38) D. Anderson, M. Desaix, M. Lisak, and M. L. Quorida-Teixeiro, J. Opt. Soc. Am. B **9**, 1358 (1992).
39) H. Bateman and A. Erdelyi, *Tables of Integral Transforms* (McGraw-Hill, New York, 1954).
40) D. R. Tilley and J. Tilley, *Superfluidity and Superconductivity* (Van Nostrand Reinhold, New York, 1974).
41) S. V. Sazonov, J. Exp. Theor. Phys. **98**, 1237 (2004).
42) S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Sov. Phys. Usp. **10**, 609 (1968).
43) S. V. Sazonov, Phys. Wave Phenom. **24**, 31 (2016).
44) A. N. Bugay and S. V. Sazonov, Phys. Rev. E **74**, 066608 (2006).
45) H. Sakaguchi and B. A. Malomed, Opt. Express **21**, 9813 (2013).
46) Yu. N. Karamzin and A. P. Sukhorukov, JETP Lett. **20**, 339 (1974).