JETP Letters, Vol. 81, No. 7, 2005, pp. 335–341. From Pis'ma v Zhurnal Éksperimental'noĭ i Teoreticheskoĭ Fiziki, Vol. 81, No. 7, 2005, pp. 419–425.

The Current Phase Relation in Josephson Tunnel Junctions¶

A. A. Golubov[1] and M. Yu. Kupriyanov[2]
[1] *Faculty of Science and Technology, University of Twente, The Netherlands*
e-mail: a.golubov@tnw.utwente.nl
[2] *Nuclear Physics Institute, Moscow State University, Moscow, 119992 Russia*
e-mail: mkupr@pn.sinp.msu.ru
Received March 4, 2005

The $J(\varphi)$ relation in SFIFS, SNINS, and SIS tunnel junctions is studied. A method for the analytical solution of linearized Usadel equations has been developed and applied to these structures. It is shown that the Josephson current across the structure has a sum of $\sin\varphi$ and $\sin 2\varphi$ components. Two different physical mechanisms are responsible for the sign of $\sin 2\varphi$. The first one is the depairing by current, which contributes positively to the $\sin 2\varphi$ term, while the second one is the finite transparency of SF or SN interfaces, which provides the negative contribution. In SFIFS junctions, where the first harmonic vanishes at the 0–π transition, the calculated second harmonic fully determines the $J(\varphi)$ curve. © *2005 Pleiades Publishing, Inc.*

PACS numbers: 74.50.+r, 74.80.Dm, 75.30.Et

It is well known that tunnel SIS Josephson junctions have a sinusoidal current–phase relation, while, with a decrease in the barrier transparency, deviations from $\sin\varphi$ take place (see [1, 2] for a review). The sign of second harmonic is important for many applications, in particular, in junctions with a more complex structure such as SNINS or SFIFS, where N is a normal metal and F is a weak metallic ferromagnet [2–4]. To analyze this problem self-consistently, one should go beyond the usual "rigid boundary conditions" (RBC) approximation.

The RBC method is an effective tool used extensively earlier for theoretical study of the proximity and Josephson effects [1, 2]. This method is based on the assumption that all nonlinear and nonequlibrium effects in a Josephson structure are located in a "weak link" connecting two superconducting electrodes. The back influence of these effects on superconductivity in the electrodes is neglected. The RBC approximation is valid if a junction has the constriction geometry. The quantitative criteria for the validity of RBC for planar SIS tunnel junctions, SS'S sandwiches, and variable thickness bridges were studied only numerically for some parameter ranges [2]. The main technical difficulty in formulating the analytical criteria of RBC validity is to find the solution to equations describing the perturbation of the superconducting state in S electrodes. In this paper, we will attack this problem by finding the solution to the linearized Usadel equations [5]. We will also use this solution to formulate the corrections to previous results obtained in the RBC approximation.

¶ This article was submitted by the authors in English.

JUNCTION MODEL

Let us consider a structure of the SFIFS type, where, for simplicity, the parameters of the SF bilayers are equal to each other. We assume that the S layers are bulk and that the dirty limit conditions are fulfilled in the S and F metals. We assume further that F metals are weak monodomain ferromagnets with a zero electron–phonon interaction constant and that the FS interfaces are not magnetically active. We will restrict ourselves to the case of parallel orientation of the exchange fields H in the ferromagnets. The results obtained for SFIFS junctions cross over to SNINS and SIS in the corresponding limits.

Under the above assumptions, the problem is reduced to the solution of the one-dimensional Usadel equations [5, 6] in S and F layers and the matching of these solutions by the appropriate boundary conditions [7]. We choose the x axis perpendicular to the plane of the interfaces with the origin at the central barrier I and introduce indexes L (left), R (right), and I for description of the materials and interface parameters of the SFIFS structure located on the left and right sides of the central barrier and at this central barrier, respectively.

The Usadel functions G and F obey the normalization condition $G_\omega^2 + F_\omega F_{-\omega}^* = 1$, which allows for the following parameterization in terms of the new function Φ:

$$G_\omega = \frac{\tilde{\omega}}{\sqrt{\tilde{\omega}^2 + \Phi_\omega\Phi_{-\omega}^*}}, \quad F_\omega = \frac{\Phi_\omega}{\sqrt{\tilde{\omega}^2 + \Phi_\omega\Phi_{-\omega}^*}}. \quad (1)$$

The quantity $\tilde{\omega} = \omega + iH$ corresponds to the general case when the exchange field H is present. However, in the S layers, $H = 0$, and we have simply $\tilde{\omega} = \omega$.

The Usadel equations [5] in the S and F layers have the form

$$\xi_S^2 \frac{\pi T_c}{\pi G_S} \frac{\partial}{\partial x}\left[G_S^2 \frac{\partial}{\partial x}\Phi_S\right] - \Phi_S = -\Delta, \quad (2)$$

$$\xi_F^2 \frac{\pi T_c}{\tilde{\omega} G_F} \frac{\partial}{\partial x}\left[G_F^2 \frac{\partial}{\partial x}\Phi_F\right] - \Phi_F = 0, \quad (3)$$

where $G_\omega = \tilde{\omega}/\sqrt{\tilde{\omega}^2 + \Phi_\omega \Phi^*_{-\omega}}$, $\tilde{\omega} = \omega + iH$ in a ferromagnet (H is the exchange field), $\tilde{\omega} = \omega$ in S and N metals, T_c and Δ are the critical temperature and the pair potential in a superconductor, $\omega = \pi T(2n + 1)$ are the Matsubara frequencies, and $\xi_{S(F)}$ are the coherence lengths related to the diffusion constants $D_{S(F)}$ as $\xi_{S(F)} = \sqrt{D_{S(F)}/2\pi T_c}$. The pair potential satisfies the self-consistency equations

$$\Delta \ln\frac{T}{T_c} + \pi T \sum_{\omega = -\infty}^{\infty} \frac{\Delta - G_S \Phi_S \operatorname{sgn}\omega}{|\omega|} = 0. \quad (4)$$

In the case of an SFIFS tunnel junction in the quasi-one-dimensional geometry, the boundary conditions at the junction plane ($x = 0$) read

$$\xi_F \frac{G_{F,L}^2}{\tilde{\omega}_L} \frac{\partial}{\partial x}\Phi_{F,L} = \xi_F \frac{G_{F,R}^2}{\tilde{\omega}_R} \frac{\partial}{\partial x}\Phi_{F,R}, \quad (5)$$

$$\gamma_{BI} \frac{\xi_F G_{FL,R}}{\tilde{\omega}_L} \frac{\partial}{\partial x}\Phi_{FL,R} = \pm G_{F,R}\left(\frac{\Phi_{F,R}}{\tilde{\omega}_R} - \frac{\Phi_{F,L}}{\tilde{\omega}_L}\right), \quad (6)$$

with

$$\gamma_{BI} = R_N \mathcal{A}_I/\rho_F \xi_F,$$

where the indices L and R refer to the left- and right-hand sides of the junction, respectively, and R_N and $\mathcal{A}_I$ are the normal resistance and the area of FIF interface.

The boundary conditions at the SF interfaces ($x = \mp d_F$) have the form [7]

$$\frac{\xi_S G_{S,k}^2}{\omega} \frac{\partial}{\partial x}\Phi_{S,k} = \gamma \frac{\xi_F G_{F,k}^2}{\tilde{\omega}_k} \frac{\partial}{\partial x}\Phi_{F,k}, \quad (7)$$

$$\pm\gamma_B \frac{\xi_F G_{F,k}}{\tilde{\omega}_k} \frac{\partial}{\partial x}\Phi_{F,k} = G_{S,k}\left(\frac{\Phi_{S,k}}{\omega} - \frac{\Phi_{F,k}}{\tilde{\omega}_k}\right), \quad (8)$$

with $\gamma_B = R_B \mathcal{A}_B/\rho_F \xi_F, \quad \gamma = \rho_S \xi_S/\rho_F \xi_F,$

where R_B and $\mathcal{A}_B$ are the resistance and the area of the SF interfaces; $\rho_{S(F)}$ is the resistivity of the S (F) layer; $k = L, R$. Both of these conditions ensure continuity of the supercurrent.

We will also suppose that, due to the low transparency of the FIF interface, the Josephson current is much smaller than the depairing current of superconducting electrodes, meaning that the suppression of superconductivity in the interior of the electrodes can be neglected and, at $x \longrightarrow \pm\infty$,

$$|\Phi_{S,k}| = \Delta_0, \quad (9)$$

where Δ_0 is the magnitude of the bulk order parameter.

LIMIT OF SMALL F LAYER THICKNESS

In this limit

$$d_F \ll \min\left(\xi_F, \sqrt{\frac{D_F}{2H}}\right) \quad (10)$$

the gradients in (3) are small and, in the second approximation of d_F/ξ_F, the solution of (3) has the form

$$\Phi_{F,k} = A_k + B_k \frac{x}{\xi_F} + \frac{x^2}{2} \frac{\tilde{\omega}_k A_k}{\pi T_c \xi_F^2 G_{F,k}}, \qquad G_{F,k}^2 = \frac{\tilde{\omega}_R^2}{\tilde{\omega}_R^2 + A_k^2(\omega)}. \quad (11)$$

Integration constants $\tilde{A}$ and $\tilde{B}$ in (11) can be found from boundary conditions at $x = 0$,

$$\frac{G_{F,L}^2}{\tilde{\omega}_L} B_L = \frac{G_{F,R}^2}{\tilde{\omega}_R} B_R = \frac{G_{F,L} G_{F,R}}{\gamma_{BI}}\left(\frac{A_R}{\tilde{\omega}_R} - \frac{A_L}{\tilde{\omega}_L}\right) \quad (12)$$

and at $x = \pm d_F$,

$$A_k = A_{0,k} \mp \gamma_B \frac{G_{F,k}}{G_{S,k} + \tilde{\omega}_k \gamma_{BM}/\pi T_c} B_k, \quad (13)$$

$$A_{0,k} = \frac{\tilde{\omega}_{R,L} \Phi_{S,k} G_{S,k}}{\omega(G_{S,k} + \tilde{\omega}_k \gamma_{BM}/\pi T_c)}, \quad \gamma_{BM} = \gamma_B \frac{d_F}{\xi_F}. \quad (14)$$

Expression (13) is valid if $\gamma_B \ll \gamma_{BI}$. Substitution of (11) and (13) into the boundary condition at $x = \pm d_F$ leads to

$$\xi_S \frac{\partial}{\partial x}\Phi_{S,k} = \pm\gamma_M \frac{G_{F,k}}{G_{S,k}^2} \frac{\omega}{\pi T_c} A_k + \gamma \frac{\omega G_{F,k}^2}{\tilde{\omega}_k G_{S,k}^2} B_k, \quad (15)$$

where $\gamma_M = \gamma d_F/\xi_F$, and we reduce boundary problem (2)–(9) to the solution of Eqs. (2), (4) in the S layers with boundary conditions (9), (15). At $H = 0$ and $(\gamma_{BI} d/\xi_F) \gg 1$, expression (15) reduces to the known result for the SN bilayer [8].

LINEARIZED USADEL EQUATIONS

Following the RBC approximation, we will start with the assumption that the suppression of superconductivity in the S layer is weak and that the solution of the Usadel equations in the superconductor has the form

$$\Phi_{S,k}(\omega) = \Delta_{0,k} + \Phi_{1,k}, \quad \Delta = \Delta_{0,k} + \Delta_{1,k}, \tag{16}$$

$$G_{S,k} = G_0 + G_{1,k}, \quad G_0 = \frac{\omega}{\sqrt{\omega^2 + \Delta_0^2}},$$
$$G_{1,k} = -\frac{G_0}{\omega^2 + \Delta_0^2}\frac{[\Delta_{0,k}^*\Phi_{1,k} + \Delta_{0,k}\Phi_{1,k}^*]}{2}, \tag{17}$$

where $\Delta_{0,k} = \Delta_0\exp\{\pm i\varphi/2 + iUx/\xi_S\}$, φ is the order parameter phase difference across the barrier, and the coefficient U describes the linear growth of the phase difference due to the supercurrent in the electrodes. Corrections to Δ_0 and $\Phi_{S,k}$ are supposed to be small:

$$|\Delta_{1,k}| \ll \Delta_0, \quad |\Phi_{1,k}| \ll \Delta_0. \tag{18}$$

The approximation is valid if the right-hand side of Eq. (15) is also small, so that

$$\xi_S\frac{\partial}{\partial x}\Phi_{1,k} = \Xi_k(\omega), \tag{19}$$

$$\Xi_k(\omega) = \pm\gamma_M\frac{\omega G_{F0,k}A_{0,k}}{\pi T_c G_0^2} + \gamma\frac{\omega G_{F0,k}^2 B_k}{\tilde{\omega}_k G_0^2},$$
$$G_{F0,k} = \frac{\omega\vartheta_k}{\sqrt{\omega^2\vartheta_k^2 + \Delta_0^2 G_0^2}}, \tag{20}$$

where $\vartheta_k = (G_0 + \tilde{\omega}_k\gamma_{BM}/\pi T_c)$ and $|\Xi(\omega)| \ll \Delta_0$. From the structure of the linearized Usadel equations boundary conditions (19), it follows that there are first-order corrections only to the magnitudes Θ and Δ_1 of functions Φ_1 and $\Delta_{1,k}$, respectively, while the phases of all of these functions coincide with those of $\Delta_{0,k}$. In this case,

$$\tilde{\Phi}_{1,k} = \Theta\exp\left\{\pm i\frac{\varphi}{2}\right\}, \quad \Delta_{1,k} = \Delta_1\exp\left\{\pm i\frac{\varphi}{2}\right\} \tag{21}$$

and, due to the symmetry of the structure, we have

$$\tilde{\omega}_R = \tilde{\omega}_L = \tilde{\omega}, \quad G_{F0,k} = G_{F0}, \quad \vartheta_k = \vartheta,$$
$$\frac{A_{0,k}}{\Delta_0} = C_0\exp\left\{\pm i\frac{\varphi}{2}\right\}, \quad C_0 = \frac{\tilde{\omega}G_0}{\omega\vartheta}, \tag{22}$$

$$\Xi_k(\omega) = \frac{G_{F0}}{G_0\vartheta}\left[\pm\gamma_M\frac{\tilde{\omega}}{\pi T_c}\cos\frac{\varphi}{2}\right.$$
$$\left.+ i\left(\gamma_M\frac{\tilde{\omega}}{\pi T_c} + 2\frac{\gamma}{\gamma_{BI}}G_{F0}\right)\sin\frac{\varphi}{2}\right]. \tag{23}$$

To write (23), we also used the fact that, in the first order with respect to $|\Xi(\omega)|$, the magnitudes of the functions $\Phi_{S,k}$ in (13) equal Δ_0 and that $G_S = G_0$.

Substituting (16), (21) into (2), (3), we arrive at the following boundary problem for Θ and Δ_1:

$$-\xi_S^2\frac{\pi T_c}{\sqrt{\omega^2 + \Delta_0^2}}\frac{\partial^2}{\partial x^2}\Theta + \Theta = \Delta_1, \tag{24}$$

$$\Delta_1\left[\ln\frac{T}{T_c} + \pi T\sum_{\omega=-\infty}^{\infty}\frac{1}{|\omega|}\right] - \pi T\sum_{\omega=-\infty}^{\infty}\frac{\omega\Theta G_0}{(\omega^2 + \Delta_0^2)} = 0, \tag{25}$$

$$\xi_S\frac{\partial}{\partial x}\Theta(\pm d_F) = \left[\mathrm{Re}\,\Xi_k(\omega)\cos\frac{\varphi}{2} \pm \mathrm{Im}\,\Xi_k(\omega)\sin\frac{\varphi}{2}\right], \tag{26}$$

$$\Xi(\pm\infty) = 0. \tag{27}$$

Due to the symmetry of the problem, it is enough to solve Eqs. (24)–(27) only in one of the electrodes, namely, for $x \geq d_F$. Using the equation for $\Delta_0(T)$,

$$\ln\frac{T}{T_c} + \pi T\sum_{\omega=-\infty}^{\infty}\frac{1}{|\omega|} = \pi T\sum_{\omega=-\infty}^{\infty}\frac{1}{\sqrt{\omega^2 + \Delta_0^2}} \tag{28}$$

and the symmetry relation $\Theta(\omega) = \Theta(-\omega)$, we can rewrite the self-consistency equation in the form

$$\Delta_1\Sigma_2 = \pi T\sum_{\omega>0}^{\infty}\frac{\pi T_c\omega^2}{(\omega^2 + \Delta_0^2)^2}\xi_S^2\frac{\partial^2}{\partial x^2}\Theta, \tag{29}$$

$$\Sigma_2 = \pi T\sum_{\omega>0}\frac{\Delta_0^2}{(\omega^2 + \Delta_0^2)^{3/2}}. \tag{30}$$

The solution of (24), (29) is

$$\Delta_1 = \sum_{\Omega>0}^{\infty}\delta_\Omega\exp\left(-q_\Omega\frac{x - d_F}{\xi_S}\right),$$
$$\Theta = \sum_{\Omega>0}^{\infty}\frac{\delta_\Omega\sqrt{\omega^2 + \Delta_0^2}}{\sqrt{\omega^2 + \Delta_0^2 - \pi T_c q_\Omega^2}}\exp\left(-q_\Omega\frac{x - d_F}{\xi_S}\right), \tag{31}$$

where the coefficients δ_Ω and q_Ω satisfy the equation

$$\Sigma_2 = \pi T\sum_{\omega>0}^{\infty}\frac{\omega^2}{(\omega^2 + \Delta_0^2)^{3/2}}\frac{q_\Omega^2\pi T_c}{\sqrt{\omega^2 + \Delta_0^2 - \pi T_c q_\Omega^2}}, \tag{32}$$

$$\sum_{\Omega>0}^{\infty}\frac{q_\Omega\delta_\Omega}{(\sqrt{\omega^2 + \Delta_0^2} - \pi T_c q_\Omega^2)} = -\frac{\Delta_0 P(\varphi, \omega)}{\sqrt{\omega^2 + \Delta_0^2}} \tag{33}$$

and $P(\varphi, \omega) = \mathrm{Re}\,\Xi_R(\omega)\cos(\varphi/2) + \mathrm{Im}\,\Xi_R(\omega)\sin(\varphi/2)$. Multiplying Eq. (33) by $\omega^2(\omega^2 + \Delta_0^2)^{-3/2}$, summing both sides of this equation over ω, and making use of (32), one can transform (33) into a system of equations for the coefficients δ_Ω, which yields

$$\delta_\Omega = -\pi T \frac{\pi T_c \Delta_0 \Omega^2 q_\Omega}{\Sigma_2 (\Omega^2 + \Delta_0^2)^2} \Lambda(\Omega, \varphi), \quad (34)$$

where

$$\Lambda(\Omega, \varphi) = \left[\gamma_M K_1(\Omega) + \frac{\gamma}{\gamma_{BI}} K_2(\Omega)(1 - \cos\varphi)\right],$$

$$K_1(\Omega) = \frac{\Omega}{\pi T_c G_0} \sqrt{\frac{\sqrt{p^2 + q^2} + p}{2(p^2 + q^2)}},$$
$$K_2(\Omega) = \frac{p G_0 + (Hq + p\Omega)\gamma_{BM}/\pi T_c}{G_0 (p^2 + q^2)}, \quad (35)$$

$$q = 2\gamma_{BM} \frac{H}{\pi T_c}\left(\gamma_{BM} \frac{\Omega}{\pi T_c} + G_0\right), \quad (36)$$

$$p = 1 + \frac{\Omega^2 - H^2}{(\pi T_c)^2} \gamma_{BM}^2 + 2 G_0 \frac{\Omega}{\pi T_c} \gamma_{BM}. \quad (37)$$

Here, $\Omega = \pi T(2m + 1)$ are the Matsubara frequencies.

As a result, the solution of boundary problem (24)–(27) has the form

$$\Delta_1 = -\pi T \sum_{\Omega > 0} \frac{\pi T_c \Delta_0 \Omega^2 q_\Omega \exp\left(-q_\Omega \frac{x - d_F}{\xi_S}\right)}{\Sigma_2 (\Omega^2 + \Delta_0^2)^2} \Lambda(\Omega, \varphi), \quad (38)$$

$$\Theta = -\pi T \sum_{\Omega > 0} \frac{\pi T_c \Delta_0 \Omega^2 q_\Omega \Lambda(\Omega, \varphi) \exp\left(-q_\Omega \frac{x - d_F}{\xi_S}\right)}{\Sigma_2 (\Omega^2 + \Delta_0^2)^2 (1 - \pi T_c q_\Omega^2 G_0/\omega)}. \quad (39)$$

In particular, at $x = d_F$, from (38) and (39) we have

$$\frac{\Theta(d_F)}{\Delta_0} = -\gamma_M \Sigma_{F1} - \frac{\gamma}{\gamma_{BI}} \Sigma_{F2} (1 - \cos\varphi), \quad (40)$$

$$\Sigma_{F1} = \pi T \sum_{\Omega > 0} \frac{\pi T_c \Omega^2 q_\Omega K_1(\Omega)}{\Sigma_2 (\Omega^2 + \Delta_0^2)^2 (1 - \pi T_c q_\Omega^2 G_0/\omega)}, \quad (41)$$

$$\Sigma_{F2} = \pi T \sum_{\Omega > 0} \frac{\pi T_c \Omega^2 q_\Omega K_2(\Omega)}{\Sigma_2 (\Omega^2 + \Delta_0^2)^2 (1 - \pi T_c q_\Omega^2 G_0/\omega)}. \quad (42)$$

To calculate sums (41) and (42), one needs to know the expression for the coefficients q_Ω, which can in general be obtained from numerical solution of Eq. (32). Since the main contribution to sums (41), (42) comes from large Ω, the asymptotic behavior of q_Ω at large Ω can be used:

$$q_\Omega^2 = \alpha \frac{\sqrt{\Omega^2 + \Delta_0^2}}{\pi T_c}, \quad \alpha = 1 - \frac{\pi T^2}{\Omega T_c} \ln \frac{\sqrt{\Omega^2 + \Delta_0^2}}{\pi T}. \quad (43)$$

The method developed is valid if the following condition is fulfilled:

$$\left(\gamma_M + \frac{\gamma}{\gamma_{BI}}\right) \max\left\{1, \ln\left[\frac{H^2 + (\pi T_c)^2}{\min\{\gamma_{BM}^2, \gamma_M^2\}(\pi T)^2}\right]\right\} \ll 1, \quad (44)$$

$$\gamma_B \ll \gamma_{BI}.$$

Therefore, for the function $\Phi_{S,k}$ in Eq. (14), we get

$$\Phi_{S,k} = (\Delta_0 + \Theta(d_F)) \exp\{\mp i\varphi/2\}, \quad (45)$$

and, substituting (45) into (13), we finally obtain

$$A_k = \left[\Delta_0 + \frac{\omega\mu C_0}{\tilde{\omega}} \Theta(d_F)\right] C_0 \exp\{\pm i\varphi/2\}$$
$$\mp 2i \frac{\gamma_B}{\gamma_{BI}} \frac{\tilde{\omega} G_0 G_{F0} \Delta_0}{\omega \vartheta^2} \sin\frac{\varphi}{2}, \quad (46)$$

$$\mu = 1 + G_0 \tilde{\omega} \gamma_{BM}/\pi T_c. \quad (47)$$

From the structure of coefficients $\tilde{A}_{R,L}$, we see that the corrections to the supercurrent across the SFIFS tunnel junction leads not only to the reduction of the critical current of the structure but also to changes in the $J_s(\varphi)$ relation.

$J_s(\varphi)$ RELATION

Using the standard expression for the supercurrent [11], boundary condition (6), and Eq. (46), we can write down the supercurrent I across the SFIFS junction in the form

$$I = (J_0 + J_{11}) \sin\varphi + J_{12} \sin 2\varphi, \quad (48)$$

where

$$J_0 = \frac{\pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0^2}{\tilde{\omega}^2 + C_0^2 \Delta_0^2}, \quad C_0 = \frac{\tilde{\omega} G_0}{\omega \vartheta}, \quad (49)$$

$$J_{11} = -\frac{2\pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0^2}{(\tilde{\omega}^2 + C_0^2 \Delta_0^2)^2} \left[\gamma_M \frac{\tilde{\omega} \omega C_0 \mu}{0} \Sigma_{F1} + \frac{\gamma_B}{\gamma_{BI}} \frac{\tilde{\omega}^2 G_{F0}}{\vartheta} + \frac{\gamma}{\gamma_{BI}} \frac{\tilde{\omega} \omega C_0 \mu}{\Delta_0} \Sigma_{F2}\right], \quad (50)$$

$$J_{12} = -\frac{\pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0^3}{(\tilde{\omega}^2 + C_0^2 \Delta_0^2)^2} \times \left[\frac{\gamma_B}{\gamma_{BI}} \frac{G_{F0} \Delta_0^2 C_0}{\vartheta} - \frac{\gamma}{\gamma_{BI}} \frac{\tilde{\omega} \omega \mu \Sigma_{F2}}{\Delta_0}\right]. \quad (51)$$

Expression (49) has been obtained previously in [9–11]. The φ-independent correction to it, J_{11}, is negative and describes the suppression of the $\sin\varphi$ component of the supercurrent. The first term in Eq. (50), which is proportional to γ_M, takes into account the suppression of superconductivity in the S electrodes due to the proximity of the thin F layer. The last two terms, which are proportional to γ_{BI}^{-1}, describe the suppression of superconductivity by the current across the junction. The larger γ_B and γ are, the weaker the superconductivity induced into the F layer and the stronger the influence of this effect.

The sign of the second harmonic J_{12} depends on the relation between γ_B and γ. At $\gamma_B = 0$, it is positive and $J(\varphi)$ relation (48) has a maximum at $\varphi = \varphi_{\max} < \pi/2$. Such a shift was predicted earlier near T_c for SIS tunnel junctions and is due to the suppression of superconductivity near the barrier by a supercurrent [12]. An increase in γ_B leads to additional phase shifts at both SF interfaces and provides the mechanism for the shift of $\varphi_{\max}$ into the region $\varphi > \pi/2$. As a result, at sufficiently large γ_B, the amplitude J_{12} changes its sign and $\varphi_{\max}$ shifts to $\varphi > \pi/2$. Such a competition between suppression by a supercurrent and by the proximity effect was first analyzed in the SNS junctions [13] at $T \approx T_c$. This fact is in full agreement with the results of numerical calculations summarized in [2].

The physical reason for different signs of J_{12} can be easily understood if we consider the two cases separately. Suppose first that γ_B is finite. In this case, the SFIFS structure may be considered a system of three Josephson junctions in series, as shown schematically in Fig. 1. For rough estimates, one can assume that the phase χ of $\Phi_{F,k}$ does not depend on ω. Demanding the equality of the currents across FIF and FS interfaces and taking into account that $I_C \propto \gamma_{BI}^{-1} \ll I_{C1} \propto \gamma_B^{-1}$ for χ, we will have

$$\chi = \varphi/2 - \frac{I_C}{I_{C1}}\sin 2\chi.$$

Substituting this χ into the expression for the supercurrent across the FIF interface, we get

$$I = I_C \sin\left(\varphi - \frac{I_C}{I_{C1}}\sin\varphi\right) \approx I_C\left(\sin\varphi - \frac{\gamma_B}{\gamma_{BI}}\sin 2\varphi\right).$$

Therefore, with increasing γ_B, the phase partly jumps at the FS interfaces, leading to a continuous crossover from the Josephson effect lumped at $x = 0$ to the phase drop distributed at $|x| \le d_F$. In full agreement with the theory of double barrier devices [2], this crossover results in the appearance of a second harmonic in $J_S(\varphi)$ with a negative sign, which provides for a maximum $J_S(\varphi)$ achieved at $\varphi \ge \pi/2$.

If $\gamma_B = 0$, the structure is always lumped at $x = 0$ and the main effect is the suppression of superconductivity by a supercurrent in the vicinity of the FIF interface, as shown schematically in Fig. 2. The resulting contribution to the full current is

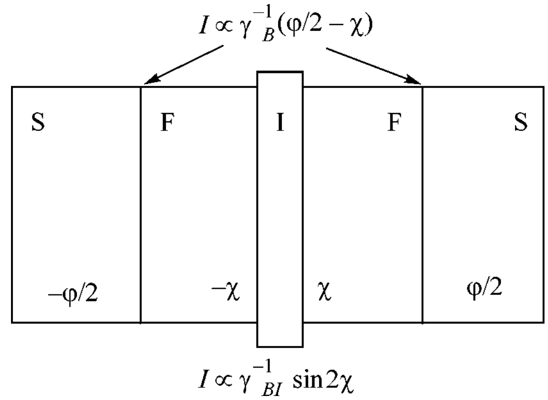

Fig. 1. The phase distribution in a SFIFS junction.

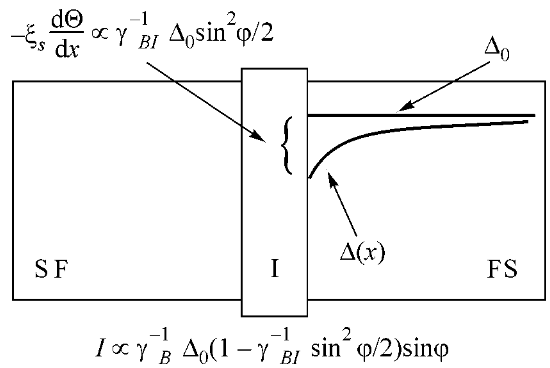

Fig. 2. Depairing by current near the tunnel barrier.

$$I_\omega \propto \gamma_{BI}^{-1}\left(\Delta_0 - \xi_S\frac{\partial\Theta}{\partial x}\right)\sin\varphi \propto \frac{\Delta_0}{\gamma_{BI}}\left(1 - \frac{\sin^2\frac{\varphi}{2}}{\gamma_{BI}}\right)\sin\varphi. \quad (52)$$

It follows directly from (52) that the amplitude of the second harmonic is positive.

The competition of the above two mechanisms of $I(\varphi)$ deformation is clearly seen from Eq. (51).

General expressions (49)–(51) can be simplified in several limiting cases.

In the symmetric SNINS tunnel junctions, $H = 0$ in both electrodes and, in the first approximation from (49), the earlier result from [8] is reproduced:

$$J_0 = \frac{2\pi T}{eR_N}\sum_{\omega \ge 0}^{\infty}\frac{\Delta_0^2}{(\omega^2 + \Delta_0^2)\Theta(\omega)},$$

$$\Theta(\omega) = (1 + 2G_0\omega\gamma_{BM}/\pi T_c + (\omega\gamma_{BM}/\pi T_c)^2)$$

while (50) and (51) reduce to

$$J_{11} = -\frac{4\pi T}{eR_N}\left[\gamma_M\Sigma_4 + \frac{\gamma_B}{\gamma_{BI}}\Sigma_5 + \frac{\gamma}{\gamma_{BI}}\Sigma_6\right],$$

$$J_{12} = -\frac{2\pi T}{eR_N}\left[\frac{\gamma_B}{\gamma_{BI}}\Sigma_7 - \frac{\gamma}{\gamma_{BI}}\Sigma_6\right],$$

where

$$\Sigma_4 = \sum_{\omega>0}^{\infty}\frac{\Delta_0 G_0\vartheta\mu\Delta_{F1}}{(\omega^2+\Delta_0^2)\Theta^2(\omega)},$$

$$\Sigma_5 = \sum_{\omega>0}^{\infty}\frac{\Delta_0^2\vartheta^2}{(\omega^2+\Delta_0^2)\Theta^{5/2}(\omega)},$$

$$\Sigma_6 = \sum_{\omega>0}^{\infty}\frac{G_0\Delta_0\vartheta\mu\Sigma_{F2}}{(\omega^2+\Delta_0^2)\Theta^2(\omega)},$$

$$\Sigma_7 = \sum_{\omega>0}^{\infty}\frac{\Delta_0^4}{(\omega^2+\Delta_0^2)^2\Theta^{5/2}(\omega)},$$

and $G_0 = \omega/\sqrt{\omega^2+\Delta_0^2}$.

In the limit $\gamma \longrightarrow 1$, H, γ_M, γ_B, $\gamma_{BM} \longrightarrow 0$, the SFIFS structure transforms into a SIS tunnel junction. In this case,

$$C_0 = 1, \quad A_{pR,L} = [\Delta_0 + \Theta(d_F)]\exp\{\pm i\varphi/2\},$$

$$\Theta(d_F) = -\frac{2}{\gamma_{BI}}\pi T\sum_{\Omega>0}\frac{\pi T_c\Delta_0\Omega^2 q_\Omega \sin^2\frac{\varphi}{2}}{\Sigma_2(\Omega^2+\Delta_0^2)^2(1-\pi T_c q_\Omega^2 G_0/\omega)},$$

and, for the supercurrent I in the first approximation, we have the well-known result of the Ambegakaokar–Baratoff theory [14]:

$$I = \frac{2\pi T}{eR_N}\sum_{\omega>0}^{\infty}\frac{\Delta_0^2}{\omega^2+\Delta_0^2}\sin\varphi.$$

Using (32) for J_{11} and J_{12}, it is easy to get

$$J_{11} = -\frac{\Delta_0}{eR_N}2\Sigma_3, \quad J_{12} = \frac{\Delta_0}{eR_N}\Sigma_3, \qquad (53)$$

$$\Sigma_3 = \frac{4}{\gamma_{BI}}\pi T\sum_{\Omega>0}\frac{\Delta_0\Omega^2}{(\Omega^2+\Delta_0^2)^2 q_\Omega},$$

and the full current across the tunnel junctions is

$$I = \frac{\Delta_0}{eR_N}\left[\frac{\pi}{2}\tanh\frac{\Delta_0}{2T} - 2\Sigma_3\right]\sin\varphi + \frac{\Delta_0\Sigma_3}{eR_N}\sin 2\varphi.$$

The critical current is achieved at a phase difference φ_c,

$$\varphi_c = \frac{\pi}{2} - \frac{4\Sigma_3}{\pi}\tanh^{-1}\frac{\Delta_0}{2T},$$

and equals

$$I_c \approx \frac{\Delta_0}{eR_N}\left[\frac{\pi}{2}\tanh\frac{\Delta_0}{2T} - 2\Sigma_3\right]$$

at $T \longrightarrow 0$, $I(\varphi)$ simplifies to

$$I_c \approx \left[\frac{\Delta_0}{eR_N}\frac{\pi}{2} - \frac{1.92}{\gamma_{BI}}\left(\frac{\pi T_c}{\Delta_0}\right)^{3/2}\right].$$

At $T \approx T_c$, Eqs. (53) transform into the result obtained in [12].

CONCLUSIONS

In summary, we have studied the current–phase relations $J_S(\varphi)$ in SFIFS, SNINS, and SIS junctions in the regime in which the second harmonic of $J_S(\varphi)$ is not small. To solve this problem self-consistently, we have developed an analytical method for solving the linearized Usadel equations. This solution describes a weak suppression of the superconducting state in a superconductor caused either by the proximity of normal or ferromagnetic material or by a current in composite SN or SF proximity systems. The method is rather general and can be applied to a wide spectrum of proximity problems.

We have demonstrated that the full current across structure (48) consists of the sum of the $\sin\varphi$ and $\sin 2\varphi$ components and have calculated the amplitudes (J_0 + J_{11}) and J_{12} of these components. In SIS and SNINS structures, the corrections J_{11} and J_{12} to the previously calculated critical current J_0 are small. The $J(\varphi)$ curve is slightly deformed so that the maximum value of the supercurrent is achieved at the phase difference φ_c, which can be smaller or larger than $\pi/2$ for a positive or negative sign of J_{12}, respectively. In SFIFS junctions, $J_0 = 0$ at the point of the transition from the 0 to the π state. This means that, in this case, the calculated values J_{11} and J_{12} determine the $J(\varphi)$ curve. Since the amplitudes J_{11} and J_{12} may have comparable magnitude, the $J(\varphi)$ measured experimentally can be essentially different from $\sin\varphi$. The validity of the approach developed is determined by inequalities (44) and $\gamma_B \ll \gamma_{BI}$. These conditions also determine the validity of rigid boundary conditions in the models [2] describing the properties of SFIFS, SNINS, and SIS tunnel junctions.

This work has been supported in part by the Russian Ministry of Education and Science, by the Russian Foundation for Basic Research grant no. 040217397-a, INTAS grant no. 01-0809, NWO-RFBR cooperation program no. 047.016.005, and the ESF PiShift program.

REFERENCES

1. K. K. Likharev, Rev. Mod. Phys. **51**, 101 (1979).
2. A. A. Golubov, M. Yu. Kupriyanov, and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).
3. V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, *et al.*, Phys. Rev. Lett. **86**, 2427 (2001).
4. S. M. Frolov, D. J. van Harlingen, V. A. Oboznov, *et al.*, Phys. Rev. B **70**, 144505 (2004).
5. K. D. Usadel, Phys. Rev. Lett. **25**, 507 (1970).
6. A. I. Buzdin and M. Yu. Kupriyanov, Pis'ma Zh. Éksp. Teor. Fiz. **53**, 308 (1991) [JETP Lett. **53**, 321 (1991)].
7. M. Yu. Kupriyanov and V. F. Lukichev, Zh. Éksp. Teor. Fiz. **94**, 139 (1988) [Sov. Phys. JETP **67**, 1163 (1988)].
8. A. A. Golubov and M. Yu. Kupriyanov, Zh. Éksp. Teor. Fiz. **96**, 1420 (1989) [Sov. Phys. JETP **69**, 805 (1989)].
9. F. S. Bergeret, A. F. Volkov, and K. B. Efetov, Phys. Rev. Lett. **86**, 3140 (2001).
10. V. N. Krivoruchko and E. A. Koshina, Phys. Rev. B **63**, 224515 (2001); Phys. Rev. B **64**, 172511 (2001).
11. A. A. Golubov, M. Yu. Kupriyanov, and Ya. V. Fominov, Pis'ma Zh. Éksp. Teor. Fiz. **75**, 223 (2002) [JETP Lett. **75**, 190 (2002)].
12. M. Yu. Kupriyanov, Pis'ma Zh. Éksp. Teor. Fiz. **56**, 414 (1992) [JETP Lett. **56**, 399 (1992)].
13. Z. G. Ivanov, M. Yu. Kupriyanov, K. K. Likharev, *et al.*, Fiz. Nizk. Temp. **7**, 560 (1981) [Sov. J. Low Temp. Phys. **7**, 274 (1981)].
14. V. Ambegaokar and A. Baratov, Phys. Rev. Lett. **10**, 486 (1963).