SPECHT'S PROBLEM FOR ASSOCIATIVE AFFINE ALGEBRAS OVER COMMUTATIVE NOETHERIAN RINGS

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ABSTRACT. In a series of papers by the authors we introduced full quivers and pseudo-quivers of representations of algebras, and used them as tools in describing PI-varieties of algebras. In this paper we apply them to obtain a complete proof of Belov's solution of Specht's problem for affine algebras over an arbitrary Noetherian ring. The inductive step relies on a theorem that enables one to find a "$\bar{q}$-characteristic coefficient-absorbing polynomial in each T-ideal Γ", i.e., a nonidentity of the representable algebra A arising from Γ, whose ideal of evaluations in A is closed under multiplication by $\bar{q}$-powers of the characteristic coefficients of matrices corresponding to the generators of A, where $\bar{q}$ is a suitably large power of the order of the base field. The passage to an arbitrary Noetherian base ring C involves localizing at finitely many elements a kind of C, and reducing to the field case by a local-global principle.

Contents

1. INTRODUCTION

Until §6, all algebras are presumed to be associative (not necessarily with unit element), over a given commutative ring C having unit element 1. The free (associative) algebra is denoted by $C\{x\}$, whose elements are called **polynomials**. The **T-ideal** of a set of polynomials **in an algebra** A is the ideal generated by all substitutions of these polynomials in A. For example, the set id(A) of polynomial identities of an algebra A is a T-ideal of $C\{x\}$. A T-ideal is **finitely based** if it is generated as a T-ideal by finitely many polynomials. For example, when A is a commutative algebra over a field of characteristic 0, id(A) is finitely based, by the single polynomial $[x_1, x_2] = x_1 x_2 - x_2 x_1$.

Our objective in this paper is to complete the affirmative proof of the affine case of Specht's problem, that any affine PI-algebra over an arbitrary commutative Noetherian ring satisfies the ACC (ascending chain condition) on T-ideals, or, equivalently, any T-ideal is finitely based. In characteristic 0 over fields, this is the celebrated theorem of Kemer [17]. When $\mathrm{char}(F) > 0$ there are nonaffine counterexamples [2, 13], with a straightforward exposition given in [9], so the best one could hope for is a positive result for affine PI-algebras. Kemer [18] proved this result for affine PI-algebras over infinite fields, and Belov extended the theorem to affine PI-algebras over arbitrary commutative Noetherian rings, in his second dissertation, with the main ideas given in [3]. We give full details of the proof (over arbitrary commutative Noetherian rings), cutting through combinatoric complications by utilizing the full strength of the theory of full quivers as expounded in [6],[7], and [8]. Actually, working over arbitrary commutative Noetherian base rings raises the question of the ACC for T-ideals of algebras that do not satisfy a PI (because the coefficients of the identities need not be invertible), but we still can obtain a positive solution in Theorem 7.6.

Note that there is no hope for such a result over a non-Noetherian commutative base ring C, because of the following observation:

Lemma 1.1. *If $\mathcal{I} \triangleleft C$, then $\mathcal{I}A$ is a T-ideal of A. In particular, $\mathcal{I}C\{x\}$ is a T-ideal of $C\{x\}$.*

Proof. Clearly $\mathcal{I}A \triangleleft A$, and it is closed under endomorphisms. $\qquad\square$

Consequently, any chain of ideals of C gives rise to a corresponding chain of T-ideals.

The positive solution to Specht's problem has structural applications, extending Braun's Theorem on the nilpotence of the radical of a relatively free algebra to the case where the base ring is Noetherian; cf. Theorem 7.12.

It might be instructive to indicate briefly where our approach differs from Kemer's characteristic 0 approach. Kemer first obtains his deep **Finite Dimensionality Theorem** that any algebra is PI-equivalent to a finite dimensional algebra A. Extending the base field, one may assume the base field K is algebraically closed, so Wedderburn's principal theorem enables one to decompose $A = \bar{A} \oplus J$ as vector spaces, where J is the radical of A and $\bar{A}$ can be identified with the algebra A/J. In two deep lemmas, exposed in [5, Section 4.4], Kemer shows that the nilpotence index of J and the vector space dimension of $\bar{A}$ over K can be described as invariants in terms of evaluations of polynomials on A, and then working combinatorically he shows that these computational invariants can be used to prove his Finite Dimensionality Theorem. Kemer's Finite Dimensionality Theorem fails for algebras over finite fields. Thus, we need some other technique, and we turn to the theory of quivers of representations of algebras into matrices, which were described computationally in [7] and [8].

Recall [5, pp. 28ff.] that an algebra A over an integral domain C is **representable** if it can be embedded as a C-subalgebra of $\mathrm{M}_n(K)$ for a suitable faithful commutative C-algebra $K \supset C$ (which can be much larger than C). In [7] we considered the **full quiver** of a representation of an associative algebra over a field, and determined properties of full quivers by means of a close examination of the structure of Zariski closed algebras, studied in [6].

The full quiver (or pseudo-quiver) is a directed, loopless graph without cycles, in which vertices correspond to simple subalgebras and edges to elements of the radical. A maximal subpath of this graph is called a **branch**. In place of Kemer's lemmas, we utilize the combinatorics of the full quiver to compute the invariants described above.

Our affirmation of Specht's problem (in the affine case) is divided into two stages: First we assume that the base ring C is a field F of order q (where q could be infinity). Recall, when $q < \infty$, that q is a power of $p = \mathrm{char}(F) < \infty$, and the Frobenius map $a \mapsto a^q$ is an F-algebra endomorphism.

In the second stage, using ring-theoretic methods, we reduce from the case of a general ring C to the situation of the first stage.

The main difficulty in this approach is to discern whether the algebras we are working with actually are representable. When the base ring is an infinite field F, Kemer [17] proved that any relatively free affine F-algebra is representable; this is also treated in [3] for F finite, but the proof is rather difficult. Consequently, we plan to treat the representability theorem in a separate paper. Although this decision enables us to provide a quicker and more transparent proof of Specht's problem, it forces us to consider T-ideals $\mathcal{I}$ for which $C\{x\}/\mathcal{I}$ need not a priori be representable. Accordingly, we need some method for "carving out" T-ideals $\mathcal{I}$ for which $C\{x\}/\mathcal{I}$ is representable.

In [8], a **trace-absorbing polynomial** for an algebra A is defined as a non-identity of A whose T-ideal is also an ideal of the algebra $\hat{A}$ obtained by taking A together with the traces adjoined. The main result of [8] was that such polynomials exist for relatively free algebras. Explicitly, we proved the following:

- Trace Adjunction Theorem ([8, Theorem 5.16]). Any branch of a basic full quiver of a relatively free algebra A naturally gives rise to a trace-absorbing polynomial of A.

The Trace Adjunction Theorem provides a powerful inductive tool. For example, as indicated in [9], it streamlines the proof of the rationality of Hilbert series of relatively free algebras.

In this paper we need to consider more generally the **characteristic coefficients** of a matrix a, by which we mean the coefficients of its characteristic polynomial $\lambda^n + \sum_{k=0}^{n-1} \alpha_k \lambda^k$. The k-th **characteristic coefficient** of a is α_k. For example, the trace and determinant are respectively the $(n-1)$-th and 0-th characteristic coefficients. In characteristic 0 one can recover all the characteristic coefficients from the traces, which is why we only dealt with traces in [8]. But here we need to generalize the result to arbitrary characteristic coefficients. Furthermore, since the multilinearization process cannot be reversed over a finite field, we cannot prove theorems about absorbing arbitrary characteristic coefficients in this case, but must content ourselves with absorbing $\bar{q}$-powers of characteristic coefficients, which we call $\bar{q}$-**characteristic coefficients**, for $\bar{q}$ a suitable power of $q = |F|$. (In fact, for technical reasons involving the quiver, we need to use an idea of Drensky [12] and consider **symmetrized** characteristic coefficients; cf. Definition 4.23.)

Thus, we generalize "trace-absorbing polynomials" to "$\bar{q}$-characteristic coefficient-absorbing polynomials" (cf. Definition 4.3) for the inductive step in the solution of Specht's problem. We do not prove here that affine PI-algebras are representable (one of the keystones of Kemer's theory in characteristic 0); this

involves a more intense study of full quivers, which we leave for a later paper. Nevertheless, once we have answered Specht's problem affirmatively for representable relatively free, affine algebras in Theorem 5.6, the passage to Noetherian base rings enables us to verify Specht's problem for all affine PI-algebras in Theorem 6.22, and an elementary module-theoretic argument yields the result for all varieties in Theorem 7.6.

Our approach parallels [8], but with an emphasis on working inside the set of evaluations of a given non-identity f of the algebra A. Although as formulated in [8, Theorem 5.16], the Trace Adjunction Theorem enables us to obtain characteristic coefficient-absorbing nonidentities; here we need to find a nonzero substitution inside f. This is done in Theorem 4.24, but at the cost of a considerably more involved proof than that of [8, Theorem 5.12]. For starters, in characteristic p, the multilinearization procedure degenerates in the sense that one cannot recover a polynomial from specializations of its multilinearizations. This means that one could have a proper inclusion of T-ideals which contain exactly the same multilinear polynomials (seen for example by taking the Boolean identity x^2+x in characteristic 2, whose multilinearization is just the identity of commutativity), so the inductive step in Specht's problem requires coping with A-quasi-linear (and A-homogeneous) polynomials rather than just with multilinear polynomials. Ironically, working in characteristic p does yield one step that is easier, which is given in Lemma 4.2.

Recall that the trace-absorbing polynomial of [8, Theorem 5.12] is obtained by means of a "hiking procedure" which forces substitutions of the polynomial into the radical. The main innovation needed here in hiking arises from the necessity to deal with several monomials of our polynomial f at a time, which was not the case in [8]. Thus, we introduce hiking of "higher stages", in particular **stage 2** hiking, which eliminates substitutions of f in the "wrong" matrix components, **stage 3** hiking, which differentiates the sizes of the base fields of the different components of maximal matrix degree, and **stage 4** hiking, which removes hidden radical substitutions.

Two definitions of actions by characteristic coefficients (one in terms of matrix computations and one in terms of polynomial evaluations) can be defined on the T-ideal that is generated by this polynomial, which thus is a common ideal of A and the algebra $\hat{A}$ obtained by adjoining traces to A, and $\hat{A}$ is Noetherian by Shirshov's Theorem [5, Chapter 2]. We perform the same reasoning for $\bar{q}$-characteristic coefficient-absorbing polynomials, but also need stage 4 hiking in Lemma 4.25, in order to identify these two actions. This enables us to pass to Noetherian algebras and conclude the verification of Specht's problem for algebras over arbitrary fields, in Theorem 5.6.

The extension to algebras over an arbitrary commutative Noetherian base ring C is given in Theorems 6.22 and 7.6. The proof has a different flavor, based on considerations about C-torsion which lead to a formal reduction to the case that C is an integral domain, in which case we repeatedly apply a version of a local-global principle and conclude by passing to its field of fractions and applying the results from the previous paragraph.

2. Preliminary material

Let us start by reviewing the background, especially about full quivers, their relationship with relatively free algebras, and the polynomials that they yield.

2.1. **Characteristic coefficients of matrices.** We start with some observations about characteristic coefficients of matrices, which we need to utilize in characteristic p.

Any matrix $a \in \mathrm{M}_n(K)$ can be viewed either as a linear transformation on the n-dimensional space $V = K^{(n)}$, and thus having Hamilton-Cayley polynomial f_a of degree n, or (via left multiplication) as a linear transformation $\tilde{a}$ on the n^2-dimensional space $\tilde{V} = \mathrm{M}_n(K)$ with Hamilton-Cayley polynomial $f_{\tilde{a}}$ of degree n^2.

Remark 2.1. The matrix $\tilde{a}$ can be identified with the matrix

$$a \otimes I \in \mathrm{M}_n(K) \otimes \mathrm{M}_n(K) \cong \mathrm{M}_{n^2}(K),$$

so its eigenvalues have the form $\beta \otimes 1 = \beta$ for each eigenvalue β of a.

Lemma 2.2. *With the notation as above, $f_{\tilde{a}} = f_a^n$ over any integral domain of arbitrary characteristic.*

Proof. By a standard specialization argument, it is enough to check the equality over the free commutative ring $\mathbb{Z}[\xi_1, \xi_2, \dots]$, which can be embedded into an algebraically closed field of characteristic 0. By Zariski density, we may assume that a is diagonal, in which case it is clear that the determinant of $\tilde{a}$ is $\det(a)^n$. But then we conclude by taking $\lambda^n - a$ instead of a. $\square$

Lemma 2.2 is often used in conjunction with the next observation.

Lemma 2.3. *Suppose $a \in \mathrm{M}_n(F)$, with $\mathrm{char}(F) = p$, and $f_a = |\lambda I - a| = \sum \alpha_i \lambda^i$ is the characteristic polynomial of a. Then, for any p-power $\bar{q}$, $\sum \alpha_i^{\bar{q}} \lambda^i$ is the characteristic polynomial of $a^{\bar{q}}$.*

Proof. Follows from $f_{a^p}(\lambda^p) = |\lambda^p I - a^p| = |\lambda I - a|^p = f_a(\lambda)^p$. $\square$

Proposition 2.4. *Suppose $a \in \mathrm{M}_n(F)$. Then the characteristic coefficients of a are integral over the F-algebra C generated by the characteristic coefficients of $\tilde{a}$.*

Proof. The integral closure $\bar{C}$ of C contains all the eigenvalues of $\tilde{a}$, which are the eigenvalues of a, so the characteristic coefficients of $\tilde{a}$ also belong to $\bar{C}$. $\square$

Definition 2.5. The $\alpha_i^{\bar{q}}$ of Lemma 2.3 are called the $\bar{q}$-**characteristic coefficients** of a.

Remark 2.6. We choose $\bar{q}$ sufficiently large so that the theory will run smoothly. By [5, Remark 2.35 and Lemma 2.36], when $\bar{q} > n$ (which is greater than the nilpotence index in the Jordan decomposition), then the matrix $a^{\bar{q}}$ is semisimple.

2.2. **Varieties of PI-algebras.** We work with polynomials in the free algebra built from a countable set of indeterminates over the given commutative Noetherian base ring C. The set $\mathrm{id}(A)$ is well known to be a T-ideal of the free algebra $C\{x\}$. More generally, given a polynomial f, we define $\langle f(A) \rangle$ to be the ideal generated by A. Thus, $\langle f(A) \rangle = 0$ iff $f \in \mathrm{id}(A)$.

A polynomial is **blended** if each indeterminate appearing nontrivially in the polynomial appears in each of its monomials. As noted in [8, Remark 22.18], any T-ideal is additively spanned by T-ideals of blended polynomials, and we only consider blended polynomials throughout this paper.

Given a T-ideal $\mathcal{I}$ of the free algebra $C\{x\}$, we can form the **relatively free algebra** $C\{x\}/\mathcal{I}$, which is free in the class of all PI-algebras A for which $\mathrm{id}(A) \subseteq \mathcal{I}$. Using this correspondence, it is enough to classify relatively free algebras.

We continue by taking our base ring to be a field F, and we investigate relatively free PI algebras in terms of the full quivers of their representations, making use of **generic elements**, as constructed in [6, Construction 7.14] and studied in [6, Theorem 7.15]. (A generic element of a finite dimensional algebra having base $\{b_1, \ldots, b_n\}$ over an infinite field is just an element of the form $\sum \xi_i b_i$, where the ξ_i are indeterminates, but the situation for algebras over a finite field becomes considerably more intricate.)

As in [8], we rely heavily on the **Capelli polynomial**

$$c_k(x_1, \ldots, x_k; y_1, \ldots, y_k) = \sum_{\pi \in S_k} \operatorname{sgn}(\pi) x_{\pi(1)} y_1 \cdots x_{\pi(k)} y_k$$

of degree $2k$ (cf. [5]). Any C-subalgebra of $\mathrm{M}_n(K)$ satisfies the identities c_k for all $k > n^2$.

Recall that a polynomial f which is linear in the first t variables is t-**alternating** if substituting $x_j \mapsto x_i$ results in 0 for any $1 \le i < j \le t$.

Definition 2.7. We denote by h_n the n^2-alternating central polynomial on $n \times n$ matrices [5, p. 25]. (We formally define $h_0 = 1$, and also have $h_1 = x_1$ and $h_2 = c_4 g$ where g is the multilinearization of the central polynomial $[y_1, y_2]^2$ for 2×2 matrices, where we use fresh indeterminates for c_4.)

When appropriate, we write $h_n(x)$ to emphasize that h_n is evaluated on indeterminates $x_1, x_2, \ldots$.

The central polynomial h_n is a crucial polynomial for our deliberations, since it is always central for $\mathrm{M}_n(C)$ regardless of the commutative base ring C.

2.3. **Quasi-linear functions and quasi-linearizations.** Although the theory works most smoothly for multilinear polynomials, in characteristic p we do not have the luxury of being able to recover a (blended) polynomial from its multilinearization, the way we can in characteristic 0. For example, one cannot recover the Boolean identity $x^2 - x$ from its multilinearization $x_1 x_2 + x_2 x_1$, which holds in any commutative algebra of characteristic 2. Thus, we must stop the linearization process before arriving at multilinear identities.

It is convenient at times to work slightly more generally with functions rather than polynomials, in order to be able to apply linear transformations.

Definition 2.8. A function f is i-**quasi-linear** on A if

$$f(\ldots, a_i + a_i', \ldots) = f(\ldots, a_i, \ldots) + f(\ldots, a_i', \ldots)$$

for all $a_i, a_i' \in A$; f is A-**quasi-linear** if f is i-quasi-linear on A for all i.

Quasi-linear polynomials are used heavily by Kemer in [18]. In contrast to [8] in which quasi-linear polynomials played a somewhat secondary role, here they are at the forefront of the theory, since we cannot avoid finite fields. Accordingly, we need to develop them here, expanding on [5, Exercise 1.9 and 1.10].

Remark 2.9. When $\operatorname{char}(F) = p$, any p^t power of an A-quasi-linear central polynomial is A-quasi-linear.

Any identity of A is obviously A-quasi-linear, since the only values are 0, so quasi-linear polynomials are only interesting for nonidentities.

Lemma 2.10. *If f is A-quasi-linear in x_i, then*

$$f(a_1, a_2, \ldots, a_{i,1} + \cdots + a_{i,d_i}, \ldots) = \sum_{j=1}^{d_i} f(a_1, a_2, \ldots, a_{i,j}, \ldots), \quad \forall a_i \in A.$$

Proof. The assertion is immediate from the definition. □

As usual, for any monomial $h(x_1, x_2, \ldots)$, define $\deg_i h$ to be the degree of h in x_i. For any polynomial $f(x_1, x_2, \ldots)$, define $\deg_i f$ to be the maximal degree $\deg_i h$ of its monomials; the sum of all such monomials is called the **leading i-part** of f.

Definition 2.11. Suppose $f(x_1, x_2, \ldots) \in C\{x\}$ has degree d_i in x_i. The i-**partial linearization** of f is

$$(1) \quad \Delta_i f := f(x_1, x_2, \ldots, x_{i,1} + \cdots + x_{i,d_i}, \ldots) - \sum_{j=1}^{d_i} f(x_1, x_2, \ldots, x_{i,j}, \ldots),$$

where the substitutions were made in the i component, and $x_{1,1}, \ldots, x_{1,d_i}$ are new variables.

Note that the i-partial linearization procedure lowers the degree of the polynomial in the various indeterminates, in the sense that the degree in each $x_{i,j}$ is less than d_i. It follows at once that applying the i-partial linearization procedure repeatedly, if necessary, to each x_i in turn in any polynomial f, yields a polynomial that is A-quasi-linear.

In [18] and [8], quasi-linearizations had been defined slightly differently, as homogeneous components of partial linearizations as defined in (1); when f belongs to a variety V defined over an infinite field, these remain in V. However, in [9, Example 2.2] we saw that over a finite field these homogeneous components might not necessarily stay in the same variety, which is the reason we have modified the customary definition.

Formally, this procedure is slightly stronger than that given in [8], but yields the following nice result:

Proposition 2.12. *Suppose $\mathrm{char}(F) = p$ and $d_i = \deg_i f$ is not a p-power. Then the leading i-part of f can be recovered from a suitable specialization of the leading k-part of an i-partial linearization of f, for suitable k.*

Proof. Taking k such that $\binom{d_i}{k}$ is not divisible by p, we note that (1) has $\binom{d_i}{k}$ terms of degree k in $x_{i,1}$ and degree $d_i - k$ in $x_{i,2}$, so we specialize $x_{i,1} \mapsto x_i$, $x_{i,2} \mapsto 1$, and all other $x_{i,j} \mapsto 0$. □

Corollary 2.13. *For any polynomial f which is not an identity of A, the T-ideal generated by f contains an A-quasi-linear nonidentity for which the degree in each indeterminate is a p-power, where $p = \mathrm{char}(F)$.*

Proof. Apply Proposition 2.12 repeatedly, until the degree in each indeterminate is a p-power. □

2.4. **Radical and semisimple substitutions.**

Remark 2.14. When studying a representation $\rho \colon A \to M_n(K)$ of an algebra A, we usually identify A with its image. In case A is an algebra over a field, we write $A = S \oplus J$, the Wedderburn decomposition into the semisimple part S and the

radical J. Then we can choose the representation such that the Zariski closure of A has the **Wedderburn block decomposition** of [6, Theorem 5.7], in which the semisimple part S is written as matrix blocks along the diagonal.

A **semisimple substitution** (into a Zariski closed algebra A) is a substitution into an element of S in some Wedderburn block of A, and a **radical substitution** is a substitution into an element of J in some Wedderburn block. A **pure substitution** is a substitution that is either a semisimple substitution or a radical substitution, i.e., into $S \oplus J$.

Definition 2.15. Write $A = S \oplus J$, the Wedderburn decomposition into the semisimple part S and the radical J. A **semisimple substitution** (into a Zariski closed algebra A) is a substitution into an element of S in some Wedderburn block of A, and a **radical substitution** is a substitution into an element of J in some Wedderburn block. A **pure substitution** is a substitution that is either a semisimple substitution or a radical substitution, i.e., into $S \oplus J$.

Remark 2.16. By [8, Remark 2.20], one can check whether an A-quasi-linear polynomial $f(x)$ is a PI of A merely by specializing the indeterminates x_i to pure substitutions.

More generally, let f be any polynomial. Given a substitution $f(\overline{x_1}; \overline{x_2}, \dots)$, if we specialize $\overline{x_1} \mapsto \overline{x_{1,1}} + \overline{x_{1,2}}$, then

$$(2) \qquad f(\overline{x_1}; \overline{x_2}, \dots) = f(\overline{x_{1,1}}; \overline{x_2}, \dots) + f(\overline{x_{1,2}}; \overline{x_2}, \dots) + \Delta f(\overline{x_{1,1}}, \overline{x_{1,2}}; \overline{x_2}, \dots),$$

where Δf is obtained from the 1-partial linearization by specializing $x_{1,j} \mapsto 0$ for all $j > 2$ and then discarding these $x_{1,j}$ from the notation.

One can interpret Equation (2) as follows:

Lemma 2.17. *Suppose that x_1 has some specialization $x_1 \mapsto \sum \overline{x_{1,j}}$ where the $\overline{x_{1,j}}$ are pure substitutions. (For example, some of them might be semisimple and others radical.) Then all specializations involving "mixing" the $\overline{x_{1,j}}$ occur in $\Delta f(\overline{x_{1,1}}, \overline{x_{1,2}}, \overline{x_2}, \dots)$.*

Proof. The "mixed" substitutions do not occur in the first two terms on the right side of Equation (2). $\qquad \square$

Lemma 2.17 enables us to apply the quasi-linearization procedure on specific substitutions of A, rather than on all of A, and will be needed when studying specific specializations of a polynomial f. If f were linear in x_1, then we could separate these into distinct specializations of f. But when f is nonlinear in x_1, we often need to turn to Lemma 2.17.

In [18], the definition of quasi-linear also included homogeneity, which can be obtained automatically over infinite fields. Here again, since we are working over finite fields, we need to be careful. We say that a function f is i-**quasi-homogeneous** of degree s_i on A if

$$f(\dots, \alpha a_i, \dots) = \alpha^{s_i} f(\dots, a_i, \dots)$$

for all $\alpha \in F, a_i \in A$, and $f(x_1, \dots, x_t; y_1, \dots, y_m)$ is A-**quasi-homogeneous** of degree s on A if f is i-quasi-homogeneous on A of degree s_i for all $1 \leq i \leq t$, with $s = s_1 \cdots s_t$.

The next lemma shows the philosophy of our approach, although we cannot use it directly because we are working over finite fields.

Remark 2.18. Suppose $f = \sum f_j \in \mathcal{I}$, where each f_j is i-quasi-homogeneous of degree $s_{i,j}$ on A. Then fixing some j_0 and taking $s = s_{i,j_0}$ yields

$$f(\ldots, \alpha a_i, \ldots) - \alpha^s f(\ldots, a_i, \ldots) = \sum_{j \neq j_0} (\alpha^{s_{i,j}} - \alpha^s) f_j(\ldots, a_i, \ldots).$$

This lowers the number of i-homogeneous components of f, and provides an inductive procedure for reducing to quasi-homogeneous functions.

Lemma 2.19. *Given any T-ideal $\mathcal{I}$ and any polynomial $f \in \mathcal{I}$ which is a nonidentity of A, we can obtain an A-quasi-homogeneous polynomial in $\mathcal{I}$.*

Proof. By Remark 2.18, taking s to be the degree of x_i in some monomial, this monomial cancels in

$$f(\ldots, \alpha a_i, \ldots) - \alpha^s f(\ldots, a_i, \ldots),$$

so one concludes by induction.   □

Definition 2.20. A specialization **radically annihilates** the polynomial f if the number of radical substitutions is at least the nilpotence index of J.

In case a substitution radically annihilates f, each monomial of f must evaluate to 0. One main idea here is that the nilpotence of the radical forces an evaluation to be 0 when the specialization radically annihilates the polynomial f.

At the outset, for full quivers defined over a field, the semisimple part S is the sum of the diagonal Wedderburn blocks of A, and J is the sum of the off-diagonal Wedderburn blocks. However, after "gluing up to infinitesimals", some of the radical J might be transferred to the diagonal blocks. For example, when A is a local algebra, there is a single block, which thus contains all of J.

Definition 2.21. A radical substitution is **internal** if it occurs in a diagonal block (after "gluing up to infinitesimals"); otherwise it is **external**.

2.4.1. *The Hamilton-Cayley equation applied to quasi-linear polynomials.* One of the key techniques used here (and throughout combinatorial PI-theory) is to absorb characteristic coefficients into some (multilinear) alternating polynomial $f(x_1, \ldots, x_t; y_1, \ldots, y_t)$, as exemplified in [5, Theorem J, p. 25]. Since we must cope with quasi-linear polynomials in this paper, we need to extend the theory to quasi-linear polynomials. Accordingly, we need another definition.

Definition 2.22. A polynomial $f(x_1, \ldots, x_t; y_1, \ldots, y_t)$ is $(A; t; \bar{q})$-**quasi-alternating** if f is A-quasi-linear in $x_1, \ldots, x_t$ and quasi-homogeneous of degree $\bar{q}$, for which f becomes 0 whenever x_i is substituted throughout for x_j for any $1 \leq i < j \leq t$.

Fortunately, the task of working with quasi-linear polynomials over infinite fields was already done in Kemer's verification of [18, Equation (40)]; he uses the terminology **forms** for our characteristic coefficients. If f is $(A; t; \bar{q})$-quasi-alternating, then we still get Kemer's conclusion. This can also be stated in the language of [5, Theorem J, Equation 1.19, page 27] (with the same proof), as follows:

$$(3) \qquad \alpha_k^{\bar{q}} f(a_1, \ldots, a_t, r_1, \ldots, r_m) = \sum f(T^{k_1} a_1, \ldots, T^{k_t} a_t, r_1, \ldots, r_m),$$

summed over all vectors $(k_1, \ldots, k_t)$ with each $k_i \in \{0, 1\}$ and $k_1 + \cdots + k_t = k$, where α_k is the k-th characteristic coefficient of a linear transformation $T : V \to V$, and f is

$(A; t; \bar{q})$-quasi-alternating. Of course, when applying (3) in arbitrary characteristic, we must consider all characteristic coefficients and not just the traces.

We want to determine a value of $\bar{q}$ for which our polynomial f will be $(A; t; \bar{q})$-quasi-alternating. When dealing with a representable affine algebra A, which has a finite number of generators, we may assume that the base field F is finite, and thus any element, viewed as a matrix in $\mathrm{M}_n(K)$, must have all of its characteristic values in a field $\bar{F}$ which is a field extension of F of some finite order $\bar{q}$. The idea is to take the characteristic polynomial of the matrix $a^{\bar{q}}$ instead of the characteristic polynomial of the matrix a.

Remark 2.23. There is a delicate issue here, insofar as Amitsur's proof of [5, Theorem J] relies on T acting on a vector space V. If we take $V = \mathrm{M}_n(K)$, then its dimension is n^2, but we can bypass this difficulty by appealing to the upcoming Lemma 2.24. (Note also that when n is a power of p, then n^2 is still a power of p. In view of Lemma 2.2, we can just replace $\bar{q}$ by $\bar{q}^2$. Likewise, we could replace T by any p-power of T.)

Lemma 2.24. *Suppose C is an algebra containing the $\bar{q}$-characteristic coefficients of a matrix $a \in \mathrm{M}_n(C)$. Then a is integral over C.*

Proof. By assumption $a^{\bar{q}}$ is integral over C, implying at once that a is integral over C. $\qquad\square$

Remark 2.25. For our applications of Shirshov's Theorem we only need to adjoin finitely many $\bar{q}$-characteristic coefficients to a given affine C-algebra A to obtain an algebra $\hat{A}$ integral of bounded degree over the commutative algebra $\hat{C}_{\bar{q}}$ obtained by adjoining the same $\bar{q}$-characteristic coefficients to C.

Thus, when we are given a representation $\rho\colon A \to \mathrm{M}_n(C)$, we stipulate that the generators of $\hat{C}_{\bar{q}}$ include all $\bar{q}$-characteristic coefficients of products of a given finite set of generators of $\rho(A)$ (viewed as a matrix in $\mathrm{M}_n(C)$).

Definition 2.26. We call $\hat{C}_{\bar{q}}$ of Remark 2.25 the $\bar{q}$-**characteristic closure** of C.

Lemma 2.27. *$\hat{C}_{\bar{q}}$ is its own $\bar{q}$-characteristic closure.*

Proof. We appeal to a result of Amitsur [1, Theorem A]; this describes the characteristic coefficients of a linear combination $\sum \beta_i r_i$ of matrices in the subalgebra generated by the characteristic coefficients of products of the r_i. $\qquad\square$

Remark 2.28. If $f(y_1, y_2, \dots)$ is A-quasi-linear in y_1 and $g(x_1, x_2, \dots)$ is $(A; t; \bar{q})$-quasi-alternating, then

$$f(g(x_1, x_2, \dots)y_1, y_2, \dots), \qquad f(y_1 g(x_1, x_2, \dots), y_2, \dots)$$

are $(A; t; \bar{q})$-quasi-alternating.

Remark 2.29 ([5, Remark G, p. 25]). Let $f(x_1, \dots, x_{t+1}; y)$ be any $(A; t; \bar{q})$-quasi-alternating polynomial. Then the polynomial

$$(4) \qquad \sum_{i=1}^{t+1} (-1)^i f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{t+1}, x_i; y)$$

is $(t + 1)$-A-quasi-alternating.

2.4.2. *Zubrilin's theory applied to quasi-linear polynomials.* Even without the trick of Remark 2.23, we could resort to a more polynomial-oriented approach developed by Zubrilin, expounded in [5], which generalizes [5, Theorem J, p. 25]. Since Zubrilin's theory as developed in [5] requires us to start with multilinear polynomials and we must cope with quasi-linear polynomials in this paper, we need to extend the theory to quasi-linear polynomials. The idea is to take the characteristic polynomial of the matrix $a^{\bar{q}}$ instead of the characteristic polynomial of the matrix a. Zubrilin's theory can be considered to be the case $\bar{q} = 1$, and extends readily to the general case. Unfortunately, the theory as developed in [5] requires many computations, so here we only indicate where the proofs are modified in this more general situation.

Recall [5, Definition 2.40] that if $f(x_1, \ldots, x_t; y)$ is multilinear in the variables x_i, then $(\delta_j f)(x_1, \ldots, x_t; y; z)$ is the sum over all the possible substitutions of zx_i for x_i in j out of the first n places. Explicitly, let $f(x_1, \ldots, x_n, \vec{y}, \vec{t})$ be multilinear in the x_i (and perhaps involving additional variables summarized as $\vec{y}$ and $\vec{t}$). Take $0 \leq k \leq n$, and expand

$$f^* = f((z+1)x_1, \ldots, (z+1)x_n, \vec{y}, \vec{t}),$$

where z is a new variable. Then we write $\delta_{k,z}^{(x,n)}(f) := \delta_{k,z}^{(x,n)}(f)(x_1, \ldots, x_n, z)$ for the homogeneous component of f^* of degree k in the noncommutative variable z.

Proposition 2.30 ([5, Corollary 2.45], [30]). *Let $f(x_1, \ldots, x_t, x_{t+1}; y)$ be any $(A; t; \bar{q})$-quasi-alternating polynomial which is linear in x_{t+1}. Also suppose the polynomial of* (4) *is an identity of A. Then A also satisfies the identity*

$$(5) \qquad \sum_{j=0}^{n} (-1)^j \delta_{j,z}^{(n)}(f(x_1, \ldots, x_n, z^{n-j}x_{n+1})) \equiv 0 \quad modulo \ \mathcal{CAP}_{n+1}.$$

We will need to use Proposition 2.30 in the general case (Theorem 6.22), even though Remark 2.23 suffices for the field-theoretic case.

3. REVIEW OF QUIVERS OF REPRESENTATIONS

Our main tool is the quiver of a representation, which we recall from [7] and [8]. (This differs from the customary definition of quiver, since it is not Morita invariant but takes into account the matrix size.)

3.1. **Full quivers and pseudo-quivers.**

Remark 3.1. Any representable algebra $A \subseteq \mathrm{M}_n(K)$ has its Wedderburn block form described in detail in [6] and [7, Definition 3.10], which is the keystone of [7]. This Wedderburn block form induces an action of A on $\mathrm{M}_n(K)$, by which we view each element $a \in A$ as a linear operator ℓ_a on $V = K^n$ via left multiplication. (Likewise, we also have a right action via right multiplication.) In the sequel, we usually consider the algebra A in this context.

For further reference, we also bring in the slightly more general notion of pseudo-quiver, to enable linear changes of basis in the representation.

See [7] and [8] for details about full quivers and pseudo-quivers. It is useful to formulate the definition purely geometrically, without reference to the original algebra.

Definition 3.2. An **(abstract) full quiver** (as well as **(abstract) pseudo-quiver**) is a directed graph Γ, without double edges and without cycles, having the following information attached to the vertices and edges:

(1) The vertices are ordered, say from **1** to **k**, and an edge can only take a vertex to a vertex of higher order. There also are identifications of vertices and of edges, called **gluing**. Gluing of vertices is of one of the following types:
  - **Identical gluing**, which identifies matrix entries in the corresponding blocks;
  - **Frobenius gluing**, which identifies matrix entries in one block with their q-th power in another block, where q is a power of p;
  - **Gluing up to infinitesimals** described in [8, Definition 2.3];
  - (in the case of pseudo-quivers) Linear relations among the vertices; cf. Remark 3.4 below.

  Each vertex is labelled with a roman numeral (I, II etc.); glued vertices are labelled with the same roman numeral.

  The first vertex listed in a glued component of vertices is also given a pair of subscripts $(n_{\mathbf{i}}, t_{\mathbf{i}})$: the **matrix degree** $n_{\mathbf{i}}$ and the **cardinality** of the corresponding field extension of F.

(2) Off-diagonal gluing (i.e., gluing among the edges) has several possible types, including **Frobenius gluing** and **proportional gluing** with an accompanying **scaling factor**. The absence of a scaling factor indicates scaling factor 1; such a gluing is called an **identical gluing** when there is no Frobenius twist indicated.

  Frobenius gluing of a block with itself and gluing up to infinitesimals can be viewed as modifying the base ring, yielding a commutative affine algebra over a field instead of a field.

  Very briefly, the **quiver of a representation** is obtained by taking the Wedderburn block form of the image, associating vertices to the diagonal blocks and arrows to the blocks above the diagonal. Gluing corresponds to identification of matrix components in the algebra. The pseudo-quiver is obtained when we make extra identifications of the vertices (which results in extra gluing).

Remark 3.3. (i) Any representation of an algebra A into Wedderburn block form gives rise to a full quiver. Starting with the k vertices $v_1, \dots, v_k$ corresponding to central idempotents of the blocks, we proceed as in Remark 3.1. We identify the entries of the respective blocks according to gluing (with identical gluing identifying entries and with Frobenius gluing corresponding to the Frobenius automorphism). For any two idempotents $e_{\mathbf{i}}, e_{\mathbf{j}}$ we choose a base of $e_{\mathbf{i}} A e_{\mathbf{j}}$ for the arrows between $\mathbf{i}$ and $\mathbf{j}$. Then by definition, any two consecutive vertices have only a single arrow joining them, although now we must accept new gluing (corresponding to linear dependence) of vertices.

(ii) Conversely, any abstract full quiver gives rise to a C-subalgebra A of $M_n(K)$ in Wedderburn block form, where we read off the diagonal blocks from the vertices (together with the matrix size, base field, and gluing), and then write down the off-diagonal parts from the arrows together with the relations that are registered together with the quiver.

Note that this observation does not require that C be a field. In this way, we can define the algebra of a quiver over an arbitrary integral domain.

Remark 3.4. Any full quiver of a representation of an algebra A gives rise to a pseudo-quiver. Starting with the k vertices $v_1, \ldots, v_k$ corresponding to central idempotents of the blocks, we proceed as in Remark 3.1. We identify the entries of the respective blocks according to gluing (with identical gluing identifying entries and with Frobenius gluing corresponding to the Frobenius automorphism). For any two idempotents $e_{\mathbf{i}}, e_{\mathbf{j}}$ we choose a base of $e_{\mathbf{i}} A e_{\mathbf{j}}$ for the arrows between $\mathbf{i}$ and $\mathbf{j}$. Then by definition, any two consecutive vertices have only a single arrow joining them, although now we must accept new gluing (corresponding to linear dependence) of vertices.

3.2. Degree vectors.

Definition 3.5. The **length** of a path $\mathcal{B}$ in a pseudo-quiver is its number of arrows, excluding loops, which equals its number of vertices minus 1. Thus, a typical path has vertices $r_1, \ldots, r_{\ell+1}$, where the vertex r_j has matrix degree n_j. We call $(n_1, \ldots, n_\ell)$ the **degree vector** of the path $\mathcal{B}$. We order the degree vectors according to the largest n_j which appears in the distinct glued components, counting multiplicity. More precisely, for any degree vector we discard any duplications (due to gluing), and then associate the number d_k to the number of components of matrix degree k. We order the degree vectors according to these sets of d_k, taken lexicographically.

For example, the degree vector $(3, 1, 3, 3)$ with no gluing of vertices is greater than $(3, 2, 2, 3, 2, 3, 3)$ with the fourth, sixth, and seventh vertices glued since 3 appears three times unglued in the first degree vector but only twice in the second degree vector.

Remark 3.6. We also define a secondary order on $\mathcal{B}$ with respect to the grade defined above, because further gluing will lower the number of elements in the grading monoid.

3.2.1. *Degenerate gluing between branches.*

Degenerate gluing is the situation in which each edge of one branch is glued to the corresponding edge of another branch; then the two branches produce the same values when we multiply out the elements in the corresponding algebra. We can eliminate degenerate gluing by passing to the pseudo-quiver, but it is often more convenient for us to make use of [8, Proposition 3.13]:

Proposition 3.7. *Any representable, relatively free algebra has a representation whose full quiver has no degenerate gluing.*

Since we are working with T-ideals, which correspond to relatively free algebras, this result enables us to bypass pseudo-quivers.

As shown in [8, Lemma 2.8], the gluing of two vertices of a pseudo-quiver can often be eliminated simply by joining the vertices (since the arrows now are linear operators). We call this process **reducing** the pseudo-quiver, and we always assume in the sequel that our pseudo-quivers are reduced.

Conversely, given a quiver or pseudo-quiver Γ, one can take an arbitrary commutative Noetherian algebra K and build a representable algebra A into $\mathrm{M}_n(K)$ from Γ. One theme of [8] is how the geometric properties of Γ yield identities and nonidentities of A.

Linear relations among the vertices of a pseudo-quiver can only occur if all paths between these vertices have the same "grade", as described in [8, §2.7]. This becomes somewhat intricate in characteristic p, in the presence of Frobenius gluing, so we review the idea here. Later on, we will need an inductive procedure which applies to when the gluing is strengthened. In principle, this is obvious, because any further gluing which does not lower the degree vector must lower the power of q used in the Frobenius twist, and so this must terminate after a finite number of steps. To state this formally requires some technical details, which we review from [8].

Take A to be the Zariski closure of a representable relatively free algebra, having full quiver Γ. We write $\mathcal{M}_\infty$ for the multiplicative monoid $\{1, q, q^2, \dots, \epsilon\}$, where $\epsilon a = \epsilon$ for every $a \in \mathcal{M}_m$. (In other words, ϵ is the "zero" element adjoined to the multiplicative monoid $\langle q \rangle$.) When the base field F is infinite, the full quiver can be separated (replacing α by $\alpha\gamma$ where $\gamma^q \ne \gamma$), and the algebra can be embedded in a graded algebra. In other words, each diagonal block is $\mathcal{M}_\infty$-graded, where the indeterminates $\lambda_{\mathbf{i}}$ are given degree 1; hence, A is naturally $\overline{\mathcal{M}}$-graded.

When F is a finite field of order q, $\mathcal{M}_m$ denotes the monoid obtained by adjoining a "zero" element ϵ to the subgroup $\langle q \rangle$ of the Euler group $U(\mathbb{Z}_{q^m-1})$, namely $\mathcal{M}_m = \{1, q, q^2, \dots, q^{m-1}, \epsilon\}$ where $\epsilon a = \epsilon$ for every $a \in \mathcal{M}_m$. Let $\overline{\mathcal{M}}$ be the semigroup $\mathcal{M}/\sim$, where $\sim$ is the equivalence relation obtained by matching the degrees of glued variables: When two vertices have Frobenius gluing $\alpha \to \phi^i(\alpha)$, we identify 1 with q^i in their respective components.

By [8, Lemma 2.7], every $\mathcal{M}_m$ is a quotient of $\mathcal{M}_\infty$, and more generally whenever $m | m'$, the natural group projection $\mathbb{Z}_{q^{m'}-1} \to \mathbb{Z}_{q^m-1}$ extends to a monoid homomorphism $\mathcal{M}_{m'} \to \mathcal{M}_m$.

The diagonal blocks S_r of A (under multiplication) can be viewed as $\mathcal{M}_{t_r}$-modules, where we define the product $[q^i]a$ to be a^{q^i}, and $[\epsilon]a = 0$. S_r itself is not graded, and we need to pass to the larger algebra B arising from the sub-Peirce decomposition of A; cf. [8, Remark 2.33].

Let $A_{r,r'}$ denote the (r, r')-sub-Peirce component of A. Then $A_{r,r'}$ is naturally a left F_r-module and right $F_{r'}$-module, so $A_{r,r'}$ is graded by the monoid $\mathcal{M}_{\hat{r}}$ obtained from $F_{\hat{r}}$, the compositum of F_r and $F_{r'}$. (In other words, if F_r has q^t elements and $F_{r'}$ has $q^{t'}$ elements, then $F_{\hat{r}}$ has $q^{\hat{t}}$ elements, where $\hat{t} = \operatorname{lcm}(t, t')$.) Since the free module is graded and the monoid $\mathcal{M}_{\hat{r}}$ is invariant under the Frobenius relations, we see that the Frobenius relations preserve the grade under $\mathcal{M}_{\hat{r}}$.

3.3. Canonization theorems.

Definition 3.8. A full quiver (resp. pseudo-quiver) is **basic** if it has a unique initial vertex r and unique terminal vertex s. A basic full quiver (resp. pseudo-quiver) Γ is **canonical** if it has vertices r' and s' satisfying the following properties:

- Γ starts with a unique path p_0 from r to r'. Thus, every vertex not on p_0 succeeds r'.
- Γ ends with a unique path p_0' from s' to s. Thus, every vertex not on p_0' precedes s'.
- Any two paths from the vertex r' to the vertex s' have the same grade.

An **enhanced canonical full quiver (resp. pseudo-quiver)** is a canonical full quiver (resp. pseudo-quiver) with uniform grade.

We recall the following "canonization theorems":

- [7, Theorem 6.12]. Any relatively free affine PI-algebra A has a representation for whose full quiver all gluing is Frobenius proportional.
- [8, Theorem 3.5]. Any basic full quiver (resp. pseudo-quiver) of a relatively free algebra can be modified (via a change of base) to an enhanced canonical full quiver (resp. pseudo-quiver).
- [8, Corollary 3.6]. Any relatively free algebra is a subdirect product of algebras with representations whose full quivers (resp. pseudo-quivers) are enhanced canonical.
- [8, Theorem 3.12]. For any C-closed T-ideal of a relatively free algebra A, the full quiver of $A' = A/\mathcal{I}$ is obtained by means of the following elementary operations on the full quiver of A: Gluing, new linear dependences on the vertices, and new relations on the base ring.

Thus, the PI-theory can be reduced to the case of enhanced canonical full quivers. Accordingly, all of the full quivers and pseudo-quivers that we consider in this paper will be enhanced canonical.

4. EVALUATIONS OF POLYNOMIALS ARISING FROM ALGEBRAS OF FULL QUIVERS

We get to the crucial point of this paper, which is how to evaluate a polynomial $f(x_1, x_2, \ldots)$ on a representable algebra A, in terms of the full quiver Γ of A. This question is quite difficult in general, but we note that the quasi-linearization of f (as defined above) is in the T-ideal generated by f, so we may assume that f is quasi-linear. Thus, by Remark 2.16, the evaluations of a quasi-linear polynomial $f(x)$ are spanned by the evaluations obtained by pure specializations of the indeterminates x_i to $S \cup J$.

The reader should already note that all of the proofs of this section are algorithmic, involving only a finite number of steps. This observation will be needed below in the reduction of the base ring from an integral domain to a field, in the second proof of Theorem 6.22.

Remark 4.1. Any nonzero evaluation arises from a string of substitutions $x_i \mapsto \overline{x_i}$ to elements corresponding to some path of the full quiver Γ. (We are permitted to have substitutions repeating in the same matrix block, i.e., the vertex repeats via a loop.) The $\overline{x_i}$ connect two vertices, say of matrix degree n_i and n_i'. Suppose t is the nilpotence index of the radical J. Then any string involving t radical substitutions is 0. If we replace x_i by $h_{n_i} x_{i,1} \cdots x_{i,t} h_{n_i} x_i$, then we still get the same evaluation when $x_{i,1}, \cdots, x_{i,t}$ are specialized to the identity matrices in $n_i \times n_i$ matrix blocks, which in particular are semisimple substitutions. Note that the h_{n_i} were inserted in order to locate the semisimple substitutions of the $x_{i,1}$ inside matrix blocks of size at least $n_i \times n_i$.

On the other hand, the number of radical substitutions must be at most the nilpotence index of A, so at least one of these extra substitutions must be semisimple, if we are still to have a nonzero evaluation. By taking $h_{n_i} x_{i,1}^{\bar{q}} x_{i,2} \cdots x_{i,t} h_{n_i} x_i$ ($\bar{q}$ as in Remark 2.6), we force this radical substitution to come from $\overline{x_{i,1}}^{\bar{q}}$.

Thus, this process eventually will yield a polynomial having a nonzero specialization corresponding to a path whose degree vector involves a semisimple substitution at each matrix block.

4.1. **Characteristic coefficient-absorbing polynomials inside T-ideals.** The main goal of [8] was to show that any relatively free representable algebra A has a T-ideal in common with a Noetherian algebra $\tilde{A}$ which is a finite module over a commutative affine algebra when A is affine. This T-ideal yields a nonidentity of a branch of A. The method was to define some action of characteristic coefficients (i.e., coefficients of the characteristic polynomial) of elements of a Zariski-closed algebra A, such that the values of the nonidentity obtained from its full quiver are closed under multiplication by these characteristic coefficients. This enabled us to preserve Hamilton-Cayley type properties in the evaluations of diagonal blocks. Here we use the same techniques, but need to refine them in order to obtain the desired polynomial within a given T-ideal obtained from an arbitrary polynomial (not necessarily arising from a branch).

Because we are working with quasi-linear polynomials instead of multilinear polynomials, we must utilize only $\bar{q}$-characteristic coefficients instead of all characteristic coefficients. Let us recall (see for example, [8, Lemma 5.1]):

Lemma 4.2. *Suppose A is a representable algebra over a field of characteristic p. When, for some m, $\bar{q} = p^m > n$ (which is greater than the nilpotence index of the Jacobson radical), then the element $a^{\bar{q}}$ is semisimple for every $a \in A$.*

This was enough to reduce various problems in [8] and [9] to the characteristic 0 case, effectively replacing $\bar{q}$ by 1, but does not suffice here to reduce Specht's problem to the characteristic 0 case. Nevertheless, it is still a key element of the proof when we keep track of $\bar{q}$.

Again, our objective is to apply the celebrated theorem of Shirshov [5, Chapter 2] to adjoin $\bar{q}$-characteristic coefficients to A and obtain an algebra finite as a module over a commutative affine F-algebra. For technical reasons, we only succeed in adjoining $\bar{q}$-powers of characteristic coefficients, so we formulate the following definition, modified from [8]:

Definition 4.3. Given a quasi-linear polynomial $f(x; y)$ in indeterminates labelled x_i, y_i, we say f is $\bar{q}$-**characteristic coefficient-absorbing** with respect to a full quiver $\Gamma = \Gamma(A)$ if the following properties hold:

   (1) f specializes to 0 under any substitution in which at least one of the x_i is specialized to a radical element of A. (In other words, the only nonzero values of f are obtained when all substitutions of the x_i are semisimple.)
   (2) $f(\mathcal{A}(\Gamma))^+$ absorbs multiplication by any $\bar{q}$-characteristic coefficient of any element in a simple (diagonal) matrix block of $\mathcal{A}(\Gamma)$.

There are two ways of obtaining intrinsically the coefficients of the characteristic polynomial

$$f_a = \lambda^n + \sum_{k=1}^{n-1} (-1)^k \alpha_j(a) \lambda^{n-k}$$

of a matrix a. Fixing k, we write α for α_k. (For example, if $k = 1$, then $\alpha(a) = \mathrm{tr}(a)$.)

Recall from [8, Definition 2.23] that we defined

$$(6) \qquad\qquad \mathrm{tr}_{\mathrm{mat}}(a) = \sum_{i,j=1}^{n} e_{ij} a e_{ji},$$

called the **matrix definition** of trace. We need a generalization.

Definition 4.4. In any matrix ring $\mathrm{M}_n(W)$, we define

$$(7) \qquad \alpha_{\mathrm{mat}}(a) := \sum_{j=1}^{n} \sum e_{j,i_1} a e_{i_2,i_2} a \cdots a e_{i_k i_k} a e_{i_1,j},$$

the inner sum taken over all vectors $(i_1, \ldots, i_k)$ of length k.

Of course, these characteristic coefficients $\alpha_{\mathrm{mat}}(a)$ commute iff W is a commutative ring. This is a key issue that we will need to address.

Lemma 4.5. *For any $\mathrm{M}_n(F)$-quasi-linear polynomial $f(x_1, x_2, \ldots)$ which is also $\mathrm{M}_n(F)$-quasi-homogeneous of degree $\bar{q}$ in x_1, the polynomial*

$$\hat{f} = f(c_{n^2}(y) x_1 c_{n^2}(z), x_2, \ldots)$$

is $\bar{q}$-characteristic coefficient absorbing in x_1.

Proof. We use the same proof as in [5, Theorem J, Equation 1.19, page 27], when the assertion is formulated as

$$(8) \qquad \alpha_k^{\bar{q}} f(a_1, \ldots, a_t, r_1, \ldots, r_m) = \sum f(T^{k_1} a_1, \ldots, T^{k_t} a_t, r_1, \ldots, r_m),$$

summed over all vectors $(k_1, \ldots, k_t)$ with each $k_i \in \{0, 1\}$ and $k_1 + \cdots + k_t = k$, where α_k is the k-th characteristic coefficient of a linear transformation $T : V \to V$, and f is $(A; t; \bar{q})$-quasi-alternating. $\qquad \square$

Remark 4.6. With notation as in (8), the Cayley-Hamilton identity for $n_i \times n_i$ matrices which are evaluations of f is

$$0 = \sum_{k=0}^{n_i} \alpha_k^{\bar{q}} f(a_1, \ldots, a_t, r_1, \ldots, r_m) = \sum_{k_1, \ldots, k_t} f(T^{k_1} a_1, \ldots, T^{k_t} a_t, r_1, \ldots, r_m),$$

which is thus an identity in the T-ideal generated by f.

Note that this is the same argument as used by Zubrilin in the proof of Proposition 2.30.

Iteration yields:

Proposition 4.7. *For any polynomial $f(x_1, x_2, \ldots)$ quasi-linear in x_1 with respect to a matrix algebra $\mathrm{M}_n(F)$, there is a polynomial $\hat{f}$ in the T-ideal generated by f which is $\bar{q}$-characteristic coefficient absorbing.*

Definition 4.8. Fixing $0 \le k < n$, we denote the k-th $\bar{q}$-characteristic coefficient of a, defined implicitly in Lemma 4.5, as $\alpha_{\mathrm{pol}}^{\bar{q}}(a)$. (Strictly speaking, k should be included in the notation, but since k is taken arbitrarily in our results, we do not bother to specify it.)

Definition 4.9. We call the identity $\sum_{k_1, \ldots, k_t} f(T^{k_1} x_1, \ldots, T^{k_t} x_t, r_1, \ldots, r_m)$, obtained in Remark 4.6, the **Hamilton-Cayley identity induced by** f.

The following result holds for arbitrary algebras of paths.

Remark 4.10. We need an action of matrix characteristic coefficients (computed on the diagonal components of the given representation of A) on the T-ideal of f. To do this, one computes the characteristic coefficient as in Definition 4.4, and applies this on each (glued) matrix component, i.e., the Peirce component corresponding to vertices on each side of an arrow. More precisely, suppose we have two Peirce

components, whose idempotents are $e_r = \sum_k e_{r,k}$ and $e_s = \sum_\ell e_{s,\ell}$. For any arrow α from (nonglued) vertices r_i to s_ℓ, we consider the matrix (a_{uv}) corresponding to α, and take characteristic coefficients on the r_k-diagonal component on the left, and the s_ℓ-diagonal component on the right. In other words, if the vertex corresponding to r has matrix degree n_i, taking an $n_i \times n_i$ matrix w, we define $\alpha^{\bar{q}}_{\mathrm{pol}_u}(w)$ as in the action of Lemma 4.5 and then the left action

$$\text{(9)} \qquad\qquad a_{u,v} \mapsto \alpha^{\bar{q}}_{\mathrm{pol}_u}(w) a_{u,v}.$$

Likewise, for an $n_j \times n_j$ matrix w we define the right action

$$\text{(10)} \qquad\qquad a_{u,v} \mapsto a_{u,v} \alpha^{\bar{q}}_{\mathrm{pol}_v}(w).$$

(However, we only need the action when the vertex is nonempty; we forego the action for empty vertices.)

We can proceed further whenever these two $\bar{q}$-characteristic coefficient actions coincide on the T-ideal of f.

Lemma 4.11. $\alpha_{\mathrm{mat}}(a)^{\bar{q}} = \alpha^{\bar{q}}_{\mathrm{pol}}(a)$ *in* $\mathrm{M}_n(C)$ *for* C *commutative.*

Thus, left multiplication by $\alpha_{\mathrm{mat}}(a)$ *acts on the set of evaluations of any* n_i^2-*alternating polynomial* $f(x; y)$ *on an* $n_i \times n_i$ *matrix component.*

Proof. Follows at once from Equation (8). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Identification of matrix actions for unmixed substitutions.** Our main objective is to introduce $\bar{q}$-characteristic coefficient-absorbing polynomials corresponding to all canonical full quivers, in order to identify these two notions of $\bar{q}$-characteristic coefficients, working with matrix substitutions inside a given polynomial. We have the natural bimodule action of $\bar{q}$-characteristic coefficients on A given in terms of the full quiver, which we can identify with $\alpha^{\bar{q}}_{\mathrm{pol}}(a)$, defined in Equation (8), whenever the matrix characteristic coefficients commute. The theory subdivides into two cases:

- The substitution of an indeterminate to sums of elements in the same glued Wedderburn component.
- The substitution of an indeterminate to sums of elements in the different glued Wedderburn components.

The techniques are different. We start with the first sort of situation, which we call "unmixed", which can be treated via the argument of Lemma 4.11. For the second sort of situation, which we call "mixed", Lemma 2.17 is applicable, but requires an intricate "hiking procedure" (defined presently) on the quasi-linearization of a polynomial.

Remark 4.12. The argument of Lemma 4.11 holds for a single diagonal matrix component over a commutative ring.

Recall that the T-**space** of a polynomial f on an F-algebra A is defined as the F-subspace of A spanned by the evaluations of f on A. Ironically, the sophisticated hiking procedure fails to handle the unmixed case since it relies on a T-space argument which thus must fail in view of Shchigolev's counterexample for ACC for T-spaces [26]. So the theory actually requires separate treatment of the "degenerate" unmixed case. The proof of Remark 4.12 involves the full force of T-ideals rather than T-spaces.

4.3. **Hiking.** We arrive at the main new idea of this paper. Let A be a Zariski closed algebra, and let f be a quasi-linear nonidentity. The goal is to replace f by a better structured nonidentity in its T-ideal, for which the $\bar{q}$-characteristic coefficients of the matrix blocks defined in components of the full quiver commute with each other and also with radical substitutions of arrows connecting glued vertices. This enables us to compute these $\bar{q}$-characteristic coefficients in terms of polynomial evaluations. We must cope with the possibility that our semisimple substitution has been sent to the 'wrong' component, either because its matrix degree is too large or the base field is of the wrong size.

We write $[a, b]$ for the additive commutator $ab - ba$, and $[a, b]_q$ for the **Frobenius commutator** $ab - b^q a$.

Lemma 4.13. *If $f(x_1, \ldots, x_n)$ is any polynomial quasi-linear in x_i, then*

$$(11) \quad f(a_1, \ldots, [a, a_i], \ldots, a_n) = f(a_1, \ldots, aa_i, \ldots, a_n) - f(a_1, \ldots, a_i a, \ldots, a_n),$$

and, more generally,

$$(12) \quad f(a_1, \ldots, [a, a_{i_1} \cdots a_{i_k}], \ldots, a_n) = \sum_{j=1}^{k} f(a_1, \ldots, a_{i_1} \cdots [a, a_{i_j}] \cdots a_{i_k}, \ldots, a_n)$$

for all substitutions in A.

Proof. By quasi-linearity, we may assume that f is a monomial, in which case we see that all of the intermediate terms cancel. $\square$

We recall the hiking procedure of [8, Lemma 5.8], but the hiking procedure here is quite subtle, and requires four different stages.

4.3.1. *Stage 1 hiking.*

Lemma 4.14. *Suppose a quasi-linear nonidentity f of a Zariski closed algebra A has a nonzero value for some semisimple substitution of some x_i in A, corresponding to an arrow in the full quiver whose initial vertex is labelled by (n_i, t_i) and whose terminal vertex is labelled by (n_i', t_i'). Replacing x_i by $[x_i, h_{n_i}]$ (where the h_{n_i} involve new indeterminates) yields a quasi-linear polynomial*

$$(13) \qquad\qquad f(\ldots, [x_i, h_{n_i}], \ldots)$$

in which any substitution of x_i into this diagonal block yields 0.

Proof. The evaluations of h_{n_i} in the semisimple part are central; hence, any nonzero value in $f(\ldots, [x_i, h_{n_i}], \ldots)$ forces us into a radical substitution. $\square$

Lemma 4.15. *For f as in Lemma* 4.14,

$$(14) \qquad\qquad \nabla_i f := f(\ldots, [x_i, h_{\max\{n_i, n_{i'}\}}], \ldots)$$

also does not vanish on A. In the case of Frobenius gluing $x \mapsto x^{q^\ell}$, we need to take instead the substitution

$$x_i \to f(\ldots, [x_i, h_{\max\{n_i, n_{i'}\}}]_{q^\ell}, \ldots).$$

Proof. There are substitutions in the appropriate diagonal block (the one whose degree is $\max\{n_i, n_{i'}\}$) for which $h_{\max\{n_i, n_{i'}\}}$ is a nonzero scalar, and we specialize x_i to an element which passes from one block to the other. $\square$

Corollary 4.16. *Any nonidentity can be hiked via successive stage* 1 *hiking to ensure semisimple substitutions in each matrix component.*

Proof. Once we have a nonzero substitution of f with external radical substitutions in all the hiked positions, we may continue to hike as much as we want without affecting the fact that we have a nonzero substitution, i.e., that f is a nonidentity.
□

Example 4.17. Stage 1 hiking is illustrated via the full quiver given for the Grassmann algebra on two generators:

(15)

Clearly the critical nonidentity for each branch is $[x_1, x_2]$, and we get the Grassmann identity $[[x_1, x_2], x_3]$ by taking $f = x_1$ and hiking.

We call this procedure (application of (14)) **stage 1** hiking, since we also need other forms of hiking which we call **stage 2**, **stage 3** and **stage 4** hiking, to be described below.

Remark 4.18. Stage 1 hiking absorbs all internal radical substitutions (cf. Definition 2.21) because of the use of the central polynomial $h_{\max\{n_i, n_{i'}\}}$, so when working with fully hiked polynomials, we need consider only the Peirce decomposition (and not the more complicated sub-Peirce decomposition; see [6]). In this manner, stage 1 hiking leads us to external radical substitutions for x_i, say from a block of degree n_i to a block of degree n_{i+1}.

Explicitly, after stage 1 hiking, we have obtained expressions of the form

(16) $g_i(x, y, z) = z_{i,1}[h_{\max\{n_i, n_{i'}\}}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2}.$

To simplify notation, we assume $n_{i+1} = n_i'$. Now we define

(17) $\tilde{f} = f(h_{n_1}, g_1, h_{n_2}, g_2, \cdots, g_\ell, h_{n_\ell+1}),$

where different indeterminates are used in each polynomial, in which we get the term

(18) $h_{n_1} g_1 h_{n_2} g_2 \cdots g_\ell h_{n_\ell+1}.$

Since the radical of a Zariski closed algebra A is nilpotent, we can perform stage 1 hiking on f only at a finite number of different positions (bounded by the nilpotence index of A) before getting an identity. Stopping before the last such hike gives us a nonidentity which would become an identity after any further hike of stage 1.

Unfortunately, our polynomial f has several different monomials, and when we hike with respect to one of these monomials, some other monomial will give us some permutation of (18), in which the substitutions might go into the "wrong component". The difficulty that we will encounter is that any matrix component can be embedded naturally into a larger matrix component, so a given matrix substitution could be viewed as being in this larger component, thereby ruining our attempts to compute with polynomials on each individual matrix component. Indeed, even a radical substitution could be replaced by a semisimple substitution in a larger component.

4.3.2. *Stage 2 hiking.* In view of the previous paragraph, we also need a second stage of hiking, to take care of substitutions into the "wrong" component.

Given a nonzero specialization of a monomial of f under the substitutions $x_i \mapsto \overline{x_i}$, $i \geq 1$, where $\overline{x_i} \in \mathrm{M}_{n_i}(K)$, consider the specialization of another (permuted) monomial of f under the substitutions $x_i \mapsto \overline{x_i}'$, $i \geq 1$, where $\overline{x_i}' \in \mathrm{M}_{n_j}(K)$ (and perhaps $j \neq i$).

Example 4.19. Consider the algebra

$$\left\{ \begin{pmatrix} \alpha & * & * & * \\ 0 & \beta & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} : \alpha, \beta \in F \right\},$$

where $*$ denotes an arbitrary element in K. The corresponding full quiver $I_{(1,1)} \to II_{(1,1)} \to III_2$ would normally give us the polynomial

$$z_{1,1}[x_{1,1}, y_1]z_{1,2}z_{2,1}[h_2(x_{2,1}, \dots), y_2]z_{3,1},$$

which could be condensed to $[x_{1,1}, y_1]z[h_2(x_{2,1}, \dots), y_2]$ since various indeterminates can be specialized to 1. But if $f(x_1, x_2, x_3, \dots)$ has both monomials $x_1 x_2 x_3 \cdots$ and $x_3 x_2 x_1 \dots$, then hiking in the second monomial yields the permuted term

$$[h_2(x_{2,1}, \dots), y_2]z[x_{1,1}, y_1]$$

which permits a nonzero evaluation with all substitutions in the lower 2×2 matrix component, and we cannot get a proper hold on the substitutions.

We need to hike f further, to guarantee that the specialization of some unintended monomial of f in (17) does not land in a subsequent matrix component $\mathrm{M}_{n_j}(K)$ for $n_j > n_i$; our next stage of hiking eliminates all such specializations.

Lemma 4.20. *Let H denote the central polynomial $h_{n_j}^{\bar{q}}$ of $M_{n_j}(K)$, and take*

$$z_{i,1}[h_{n_i}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2}g_{i+1} \cdots g_{j-1}H^{q_1}$$
$$- z_{i,1}H^{q_2}[h_{n_i}(x_{i,1}, x_{i,2}, \dots), y_i]z_{i,2}g_{i+1} \cdots g_{j-1}$$

(for each pair (q_1, q_2) that occurs in Frobenius twists in the branch; in characteristic 0 we would just take $q_1 = q_2 = 1$). The product of these terms, taken over all the q_1 and q_2, becomes nonzero iff the substitution is to the i component (of matrix size n_i).

Proof. Specializing this expression into the j-component (of size n_j) would yield two equal terms which cancel, since H takes on scalar values, and thus yield 0. But specializing into the i component (of size n_i) would yield one term nonzero and the other 0 since H is an identity on $n_i \times n_i$ matrices, so their difference would be nonzero. $\qquad \square$

In this way, we eliminate the "wrong" specializations in other monomials of f while preserving the "correct" ones. The modification of f according to this specialization is called **stage 2 hiking of the polynomial f**.

4.3.3. *Stage 3 hiking.* So far we have guaranteed the specializations to be in the matrix components of the correct size, but we need to fine tune them still further because the centers of the components may be of different sizes. Next, we will want to reduce to the case where for any two branches, the base field for each vertex has the same order. Suppose $\mathcal{B}'$ is another branch with the same degree vector. If the corresponding base fields for the i-th vertex of $\mathcal{B}$ and $\mathcal{B}'$ are n_i and n_i' respectively, we take $t_i = q^{n_i}$ and replace x_i by $(h_{n_i}^{t_i} - h_{n_i})x_i$. The modification of f according to this specialization is called **stage 3 hiking of the polynomial** f.

4.3.4. *Stage 4 hiking.* Some of the radical substitutions are **internal** in the sense that they occur in a diagonal block (after "gluing up to infinitesimals"). Hiking absorbs all internal radical substitutions, because of the use of the central polynomial h_{n_i}, so when working with fully hiked polynomials, we need consider only the Peirce decomposition (and not the more complicated sub-Peirce decomposition; cf. [6]).

Lemma 4.21. *There is a substitution to hike f further such that*

$$
(19) \qquad \tilde{c}_{n_i^2}(y)x_ic_{n_i'^2(y)}\tilde{c}_{n_i^2}(\alpha_k y)x_ic_{n_i'^2(\alpha_k y)} - \tilde{c}_{n_i^2}(\alpha_k y)x_ic_{n_i'^2(\alpha_k y)}\tilde{c}_{n_i^2}(y)x_ic_{n_i'^2(y)}
$$

vanishes under any specialization to the n_i, n_i' blocks.

Proof. By Proposition 4.7 there is a Capelli polynomial $\tilde{c}_{n_i^2}$ and p-power $\bar{q}$ such that

$$
(20) \qquad\qquad \tilde{c}_{n_i^2}(\alpha_k y)x_ic_{n_i'^2}(y) = \alpha_k^{\bar{q}}(y_1)c_{n_i^2}(y)x_ic_{n_i'^2}(y)
$$

on the diagonal blocks.

Since $\bar{q}$-characteristic coefficients commute on any diagonal block, we see from this that

$$
(21) \qquad \tilde{c}_{n_i^2}(y)x_ic_{n_i'^2(y)}\tilde{c}_{n_i^2}(\alpha_k y)x_ic_{n_i'^2(\alpha_k y)} - \tilde{c}_{n_i^2}(\alpha_k y)x_ic_{n_i'^2(\alpha_k y)}\tilde{c}_{n_i^2}(y)x_ic_{n_i'^2(y)}
$$

vanishes identically on any diagonal block, where $z = \alpha_k y$.

One concludes from this that substituting (21) for x_i would hike our polynomial one step further. But there are only finitely many ways of performing this hiking procedure. Thus, after a finite number of hikes, we arrive at a polynomial in which we have complete control of the substitutions and the $\bar{q}$-characteristic coefficients commute. $\qquad\square$

4.4. **Admissible polynomials.** Although hiking is the key tool in this analysis, one must note that the only time that the hiking procedure makes the two actions of Remark 4.10 coincide is when it is applied to the components of maximal matrix degree. In other words, the polynomial f is required to have a nonzero evaluation on a vector of maximal matrix degree. Thus we must consider the following sort of polynomial.

Definition 4.22. A polynomial f is A-**admissible** on a Zariski-closed algebra A if f takes on some nonzero evaluation on a vector of maximal matrix degree. We denote such a vector as $v_{\mathcal{B}}$, where $\mathcal{B}$ is the branch of the full quiver which gives rise to $v_{\mathcal{B}}$, and call $v_{\mathcal{B}}$ the **matrix vector of** f.

If all of the substitutions of an indeterminate are to glued edges of the full quiver, connecting vertices corresponding to the same matrix degree n_1, then the notion

of admissibility is irrelevant, since one could just replace f by $c_{n_1^2} f$ and obtain the desired action on the matrix component from Equation (8).

But there is a subtle difficulty. Without gluing, we could proceed directly via Remark 4.10. But gluing between branches leads to complications in applying Remark 4.10. For example, one could have two strings $I \to II \to III$ and $I \to III \to II$, whereby the substitutions in f go to incompatible components. The following definition, inspired by Drensky [12], enables us to bypass this difficulty in the proof of the next theorem.

Definition 4.23. Given matrices $a_1, \ldots, a_k$, the **symmetrized** $(t; j)$ characteristic coefficient is the j-elementary symmetric function applied to the t-characteristic coefficients of $a_1, \ldots, a_k$.

For example, taking $t = 1$, the symmetrized $(1, j)$-characteristic coefficients α_t are

$$\sum_{j=1}^{k} \text{trace}(a_j), \quad \sum_{j_1 > j_2} \text{trace}(a_{j_1}) \, \text{trace}(a_{j_2}), \quad \ldots, \quad \prod_{j=1}^{k} \text{trace}(a_j).$$

Theorem 4.24. *Suppose f is an A-admissible nonidentity of a representable, relatively free algebra A. Then the T-ideal $\mathcal{I}$ generated by f contains a $\bar{q}$-characteristic coefficient-absorbing A-admissible nonidentity $\tilde{f}$.*

Furthermore, the T-ideal $\mathcal{I}_{\mathcal{B}}$ of all fully hiked A-admissible polynomial obtained from the degree vector $v_{\mathcal{B}}$ is comprised of evaluations of $\bar{q}$-characteristic coefficient-absorbing polynomials, comprised of sums of evaluations on pure specializations in $\mathcal{B}$.

Proof. Let Γ be the full quiver of A. We follow the proof of [8, Theorem 5.8], where we follow the process of building the polynomial Φ of [8, Definition 4.11], but we must be more careful. Whereas in [8, Theorem 5.12] we were looking for any trace-absorbing nonidentity and thus could hike the polynomial of an arbitrary maximal path of a pseudo-quiver, now we need to work within the polynomial f belonging to a given T-ideal and thus work simultaneously with all maximal paths of the pseudo-quiver of the corresponding relatively free algebra.

Our polynomial f need not be multilinear, although we can make it A-quasi-linear with respect to A. Since the other polynomials we insert are multilinear, $\tilde{f}$ remains A-quasi-linear. But f need not even be A-quasi-homogeneous, so multiplying some variable by a $\bar{q}$-characteristic coefficient might throw f out of the T-ideal $\mathcal{I}$.

If the base field F were infinite, we could apply Lemma 2.19 to replace f by an A-quasi-homogeneous polynomial, but in general this cannot be done so easily. Accordingly, we adopt the following strategy in characteristic p:

We say that two strings in $A := A(\Gamma)$ are **compatible** if their degree vectors are the same. (This means that the matrix sizes of their semisimple substitutions match.) Our overall goal is to modify f so that all nonzero substitutions in f are compatible, and also to match the Frobenius twists, thereby enabling us to define characteristic coefficient-absorption.

- We work with symmetrized $\bar{q}$-characteristic coefficients instead of traces. (The reason is given before Definition 4.23; we take $\bar{q}$-powers to make sure that we are working with semisimple matrices.)
- We pass to $\bar{q}$ powers of transformations for a suitable power $\bar{q}$ of p.

- We pick a particular branch $\mathcal{B}$ of A on which we can place the substitutions of a nonzero evaluation of f, and modify $\tilde{f}$ so that all branches that do not have the same degree vector $v_{\mathcal{B}}$ as that of $\mathcal{B}$ (cf. Definition 3.5) become incompatible.
- We make branches incompatible to $\mathcal{B}$ when they do not have the same size base fields as $\mathcal{B}$ for the corresponding vertices.
- We eliminate all Frobenius twists which do match those of $\mathcal{B}$.

The hiking procedure only involves a finite number of polynomials, and is implemented in the following technical lemma:

Lemma 4.25 (Compatibility Lemma)**.** *For any A-admissible nonidentity f of a representable Zariski-closed algebra A, the T-ideal $\mathcal{I}$ generated by the polynomial f contains a symmetrized $\bar{q}$-characteristic coefficient-absorbing polynomial $\bar{f}$, not an identity of A, in which all substitutions providing nonzero evaluations of f are compatible.*

Proof. If all the vertices of the full quiver Γ of A are glued, we are done by Remark 4.12. Thus, we assume that Γ has nonglued vertices, and thus has strings with degree vectors of length > 1.

Since applying the hiking procedure of Lemma 4.21 does not change the hypotheses, we may assume that $\tilde{f}$ is fully hiked. We choose $\bar{q}$ according to Lemma 4.2.

We consider all substitutions $x_i \mapsto \overline{x_i}$ to semisimple and radical elements. Of all substitutions which do not annihilate f, some monomial of f then specializes to a nonzero evaluation, i.e., a path in the full quiver, and we choose such a substitution whose path $\mathcal{P}$ has maximal degree vector, and, after modifying f along the lines of Remark 4.1, we may assume that the largest component n_i of the degree vector involves some semisimple substitution which, with slight abuse of notation, we denote as $\overline{x_i}$. We want to make any other substitution $x_i \mapsto \overline{x_i'}$ compatible with our given substitution.

Suppose first that $\overline{x_i}$ is a substitution connecting vertices of matrix degree $n_{i,1}$ and $n_{i,2}$. In particular, since any two consecutive arrows have a common vertex, $n_{i,1} = \overline{x_i}$ must be an external radical substitution; if $\overline{x_i}$ is a semisimple substitution or an internal radical substitution, then $n_{i,1} = n_{i,2}$.

Take any substitution $\overline{x_i'}$, connecting vertices of degree $n_{i,1}'$ and $n_{i,2}'$. Replacing x_i by $h_{n_i} x_i h_{n_i}$ would annihilate any substitution to a component of smaller matrix degree, so we may assume that $n_{i,1}' \geq n_{i,1}$ and $n_{i,2}' \geq n_{i,2}$.

Take i such that n_i is maximal. Since the path $\mathcal{P}$ is assumed to have maximal degree vector, we must have $n_{i,1}' = n_i = n_{i,2}'$. Furthermore, replacing x_i by $x_i^{\bar{q}}$ forces the semisimple substitution $\overline{x_i'}^{\bar{q}}$.

We will force all our new substitutions $\overline{x_j'}$ to be compatible with the original ones along $\mathcal{P}$, by working around this vertex.

Inductively (working backwards), we assume that $n_k' = n_k$ for all $j < k \leq i$. We already know that $n_j' \geq n_j$, but want to force $n_j' = n_j$. It is enough to check this for any external radical substitution $\overline{x_j}$, since these fix the degree vectors.

If $\overline{x_j}'$ is also an external radical substitution, then $n_j' = n_j$ by maximality of the degree vector, so we are done unless $\overline{x_j}'$ is in the diagonal matrix block of degree $n_k \times n_k$; in other words, $n_j' = n_k$, whereas $n_j < n_k$.

Now we apply stage 2 hiking (as described in Lemma 4.20), which preserves the compatible substitutions and annihilates the incompatible substitutions.

Thus, we only need concern ourselves with substitutions $\overline{x_j}$ into the same matrix component along the diagonal.

Since f is presumed to be fully hiked, for any internal radical substitution $\overline{x_j}$, we may assume that $\overline{x_j}'$ also is an internal radical substitution; otherwise further hiking will not affect an evaluation along $\mathcal{P}$ but will make the other evaluation 0.

As noted above, when $\overline{x_j}$ is a semisimple substitution, taking $x_j^{\bar{q}}$ forces the semisimple substitution $\overline{x_j'}^{\bar{q}}$.

In conclusion, by modifying f we have forced all the substitutions $\overline{x'}$ to be compatible with the original substitutions $\overline{x}$.

Next, we want to reduce to the case where for any two branches, the base field for each vertex has the same order. Suppose $\mathcal{B}'$ is another branch with the same degree vector, but with a base field of different order. Stage 3 hiking will zero out all substitutions of x_i in $\mathcal{B}'$, and thus make $\mathcal{B}'$ incompatible.

Since by definition any further hike of $\tilde{f}$ yields an identity, Lemma 4.21 dictates that polynomial $\bar{q}$-characteristic coefficients defined in terms of f via Equation (8) must commute. There is a subtlety involved when the sub-Peirce component involves consecutively glued vertices, since then we need the radical substitution b of an arrow connecting glued vertices to commute with $\alpha_{\mathrm{pol}}^{\bar{q}}(a)$. But this is because taking any commutator hikes the polynomial further and thus is 0.

Now, as before, Lemma 4.21 dictates that polynomial $\bar{q}$-characteristic coefficients defined in terms of f via Equation (8) must commute, and thus the symmetrized $\bar{q}$-characteristic coefficients commute, and applying Lemma 4.13 shows that they commute with any radical substitutions of arrows connecting glued vertices. Thus, the polynomial action on $\mathcal{I}$ coincides with the well-defined matrix action as described in Remark 4.10.

We still might encounter two different branches with the same degree vector but with "crossover gluing", i.e., $I \to II \to III$ together with $III \to II \to I$. Applying the same polynomial f to these two different branches might produce different results. To sidestep this difficulty, assuming there are k such glued branches, we consider all possible matrices $a_1, \dots, a_k$ appearing in the corresponding position of these k branches, and take the elementary symmetric functions $\sigma_1, \dots, \sigma_k$ on their $\bar{q}$-characteristic coefficients; in other words, we take the symmetrized $\bar{q}$-characteristic coefficients, as defined above. Now the polynomial action clearly is the same on the σ_j, and thus on all symmetric functions on the characteristic coefficients of $a_1, \dots, a_k$.

Since the nonidentities all contain an n_i^2-alternating polynomial at the component of matrix degree n_i for each i, and we apply the $\bar{q}$-characteristic coefficient action simultaneously to each of these polynomials, their T-spaces are closed under multiplication by $\bar{q}$-characteristic coefficients of the simple components of semisimple substitutions, so we have a $\bar{q}$-characteristic coefficient-absorbing polynomial. □

Note that to apply the lemma we have to take Frobenius gluing into account. When substituting into blocks with Frobenius gluing, we get characteristic coefficents with different Frobenius twists, and then we do the symmetrization.

The lemma yields the proof of the first assertion of Theorem 4.24, since we have obtained the desired $\bar{q}$-characteristic coefficient-absorbing A-admissible nonidentity.

For the last assertion of the theorem, we note that the $\bar{q}$-characteristic coefficient-absorbing properties of the lemma were proved by means only of the properties of

the degree vector $v_\mathcal{B}$ and not on the specific polynomial $\tilde{f}$, and $\tilde{f}$ vanishes for all pure substitutions to branches other than $\mathcal{B}$. But $x\tilde{f}$ also is $\bar{q}$-characteristic coefficient-absorbing with respect to the same degree vector, and likewise the $\bar{q}$-characteristic coefficient-absorbing properties pass to sums and homomorphic images. $\qquad\square$

We have completed the first step in our program:

Theorem 4.26 ($\bar{q}$-Characteristic Value Adjunction Theorem)**.** *Let $\hat{A}$ be the algebra obtained by adjoining to A the matrix symmetrized $\bar{q}$-characteristic coefficients of products of the sub-Peirce components of the generic generators of A (of length up to the bound of Shirshov's Theorem* [5, Chapter 2]*), and let $\hat{C}$ be the algebra obtained by adjoining to F these symmetrized $\bar{q}$-characteristic coefficients. For any nonidentity f of a representable relatively free affine algebra A, the T-ideal $\mathcal{I}$ generated by the polynomial f contains a nonzero T-ideal which is also an ideal of the algebra $\hat{A}$.*

Also, $\hat{A}$ is a finite module over $\hat{C}$, and in particular is Noetherian.

Proof. We follow the proof of [8, Theorem 5.16]. First we apply the partial linearization procedure to make f A-quasi-linear. In view of [6, Theorem 7.20 and Corollary 7.21], we may assume that the generators of A are generic elements, say $X_1, \ldots, X_t$. We adjoin the Peirce components of these generic elements, noting that because the polynomial obtained by Theorem 4.24 is fully hiked, all substitutions involving these Peirce components involve a product of a maximal number of radical elements and thus still are in its T-ideal $\mathcal{I}$.

Let $\hat{C}'$ be the commutative algebra generated by all the characteristic coefficients in the statement of the theorem. Then $\hat{C}'$ is a finite module over $\hat{C}$, implying $\hat{A}$ is finite over $\hat{C}$, in view of Shirshov's Theorem. $\qquad\square$

Note that in the affine case we can work with finitely many entries, and thus $\hat{C}'$ is a finite module; this simple argument fails in the nonaffine case, which explains why this theory only applies to affine algebras.

We want to adjoin the matrix $\bar{q}$-characteristic coefficients by means of evaluations of the hiked polynomial $\tilde{f}$. This can be done for the largest size Peirce components of these new components (applied *simultaneously* to each generator and each Peirce component) to obtain an algebra $\hat{A}$, and we obtain a finite submodule via the following modification of Shirshov's Theorem:

Definition 4.27. Given a module M over a C-algebra A and a commutative subalgebra $C_1 \subset A$, we say that an element $a \in A$ is **integral** over C_1 with respect to M if there is some monic polynomial $f \in C_1[\lambda]$ such that $f(c)M = 0$. C_1 is **integral** with respect to M if each element of C_1 is integral with respect to M.

Theorem 4.28. *Suppose $A = C\{a_1, \ldots, a_\ell\}$ and M is an A-module, and A contains a commutative (not necessarily central) subalgebra C_1 such that each word in the generators of length at most the PI-degree is integral over C_1 with respect to M, and furthermore $(a_i c - c a_i)M = 0$ for each $1 \le i \le \ell$ and each $c \in C_1$. (In other words, $A/\mathrm{Ann}_A\, M$ is a C_1-algebra in the natural way.) Then $A/\mathrm{Ann}\, M$ is finite as a C_1-algebra.*

Proof. Apply Shirshov's Height Theorem [5, Theorem 2.3] to $A/\mathrm{Ann}_A\, M$. $\qquad\square$

Inspired by [8, Theorem 3.11], we formulate the following definition.

Definition 4.29. Suppose Γ is the full quiver of an algebra A. A **reduction** of Γ is a pseudo-quiver Γ' obtained by at least one of the following possible procedures:

(1) New relations on the base ring and its pseudo-quiver Γ' obtained by appropriate new gluing. This means:
- Gluing, perhaps up to infinitesimals, with or without a Frobenius twist (when the gluing is of a block with itself, with a Frobenius twist, it must become finite).
- New quasi-linear relations on arrows, perhaps up to infinitesimals.
- Reducing the matrix degree of a block attached to a vertex.

(2) New linear dependences on vertices (which could include canceling extraneous vertices) between which any two paths must have the same grade.

A **subdirect reduction** $\{\Gamma'_1, \ldots, \Gamma'_m\}$ of Γ is a finite set of reductions of Γ. A quiver is **subdirectly irreducible** if it has no proper subdirect reduction.

Lemma 4.30. *Every descending chain of reductions of our original pseudo-quiver must terminate after a finite number of steps.*

Proof. By definition, any reduction erases or identifies vertices or arrows (after sufficiently many new quasi-linear relations), or lowers the degree vectors of the branches lexicographically, or lowers their grades (Remark 3.6). Each of these processes must terminate, so the reduction procedure must terminate. □

This is the key to our discussion of Specht's problem, since it enables us to formulate proofs by induction on the reduction of a pseudo-quiver.

Lemma 4.31. *Suppose the algebra A and the polynomial $\bar{f}$ are as in Lemma 4.25. Then the full quiver Γ' corresponding to the T-ideal $\operatorname{id}(A) \cup \{\bar{f}\}$ is a reduction of the full quiver Γ corresponding to the $\operatorname{id}(A)$.*

Proof. By construction, $\bar{f}$ has nonzero evaluations along the algebra of Γ, so the full quiver Γ' could not be Γ, and thus must be a reduction. □

Remark 4.32. In summary, given a T-ideal $\mathcal{I}$, the Zariski closure A of its relatively free algebra $F\{x\}/\mathcal{I}$ has some full quiver Γ. Any A-admissible non-identity f gives rise to a symmetrized $\bar{q}$-characteristic coefficient-absorbing polynomial $\bar{f}$, not an identity of A. Letting $\mathcal{I}'$ be the T-ideal generated by $\mathcal{I} \cup \{\bar{f}\}$, we see that the full quiver Γ' of the Zariski closure A of the relatively free algebra $F\{x\}/\mathcal{I}'$ is a reduction of Γ.

5. SOLUTION OF SPECHT'S PROBLEM
FOR AFFINE ALGEBRAS OVER FINITE FIELDS

Our verification of Specht's problem over finite fields involves an inductive procedure on full quivers. After getting started, we need each chain of reductions of a full quiver to terminate after a finite number of steps. To do so, we must cope with infinitesimals, which appear in Definition 4.29, requiring a few observations about Noetherian modules in order to wrap up the proof. In all of the applications here, the Noetherian module will be a relatively free associative algebra, but for future applications to nonassociative algebras we rely only on the module structure and state these basic observations without referring to the algebra multiplication. Since Specht's problem is solved by Kemer in characteristic 0, we assume throughout this section that the base field has characteristic p.

5.1. **Torsion over fields of characteristic p.** Throughout this subsection, we assume that F is a field of characteristic p, and C is a commutative Noetherian F-algebra.

Definition 5.1. Let M be a C-module. For any $a \in M$, an element $c \in C$ is a-**torsion** if there is $k > 0$ such that $c^k a = 0$. An element $c \in C$ is M-**torsion** if it is a-torsion for each $a \in M$. We define $\mathrm{tor}(C)_a = \{c \in C : c \text{ is } a\text{-torsion}\}$.

Lemma 5.2. *For any finite C-module M and any $a \in M$, $\mathrm{tor}(C)_a$ is an ideal of C. Furthermore, define $\mathrm{tor}(C)_{a;k} = \{c \in \mathrm{tor}(C)_a : c^{p^k} a = 0\}$. Then $\mathrm{tor}(C)_a = \mathrm{tor}(C)_{a;k}$ for some k.*

Proof. $\mathrm{tor}(C)_a$ is an ideal, since we are in characteristic p. Then the series

$$\mathrm{tor}(C)_{a;1} \subseteq \mathrm{tor}(C)_{a;2} \subseteq \cdots$$

stabilizes, so $\mathrm{tor}(C)_a = \mathrm{tor}(C)_{a;k}$ for some k. $\square$

Proposition 5.3. *Suppose $\hat{A} = \hat{C}\{a_1, \ldots, a_t\}$ is a relatively free, affine algebra over a commutative Noetherian F-algebra $\hat{C}$. Then $\hat{A}$ is a finite subdirect product of an algebra $\hat{A}'$ defined over the $\hat{C}/\mathrm{tor}(\hat{C})_{a_i}$, $1 \le i \le t$, together with the $\left\{ \hat{A}/c^j\hat{A} : c \in \hat{C}, \ j < k \right\}$ where k is the maximum of the torsion indices of $a_1, \ldots, a_t$.*

Proof. Let $\hat{A}_i$ denote the direct product of the localizations of $\hat{A}$ at the (finitely many) minimal prime ideals of $\mathrm{Ann}_C a_i$. There is a natural map

$$\phi : \hat{A} \to \bigoplus \hat{A}_i \oplus \left(\bigoplus_{c \in \hat{C}, \ j < k} \hat{A}/c^j\hat{A} \right).$$

If $a \in \ker\phi$, then looking at the first component we see that a is annihilated by some power of some $c \in \hat{C}$, but this is preserved in one of the other components of $\phi(\hat{A})$; hence, $a = 0$. In other words, ϕ is an injection. $\square$

We quote [8, Lemma 3.10], in order to continue:

Suppose A is a relatively free PI-algebra with pseudo-quiver Γ with respect to a representation $\rho : A \mapsto \mathrm{M}_n(C)$, and $\mathcal{I} = \mathrm{id}(A)$ is a C-closed T-ideal. Then A is PI-equivalent to the algebra of the pseudo-quiver Γ.

5.2. **The main theorem in the field-theoretic case.** We are ready to solve Specht's problem for affine algebras over finite fields. Let us recall a key result from [5, Corollary 4.9].

Proposition 5.4. *Any T-ideal of an affine F-algebra contains the set of identities of some finite dimensional algebra, and thus of $\mathrm{M}_n(F)$ for some n.*

(The proof is characteristic free: The radical is nilpotent by the theorem of Braun-Kemer-Razmyslov, so one can display the relatively free algebra A as a homomorphic image of a generalized upper triangular matrix algebra, by a theorem of Lewin, which satisfies the identities of $n \times n$ matrices.)

Lemma 5.5. *Suppose A is a relatively free affine algebra in the variety of a Zariski closed algebra B.*

Consider a maximal path in the full quiver of B with the corresponding degree vector v_A. Let $\mathcal{J}$ be the ideal generated by the homogeneous elements of the degree

vector v_A. Then $A/\mathcal{J}$ is the relatively free algebra of a Zariski closed algebra, and hence representable, and its full quiver has fewer maximal paths of degree v_A than A.

Proof. The proof is similar to that of the Second Canonization Theorem, [8, Theorem 3.7]. Consider a maximal graded component in A. Add characteristic coefficients of the generators of the generic algebra constructed from B, and note that they agree with the grading of the paths. Factoring out the product corresponding to the maximal degree vector we obtain a representable algebra, B'. Construct the full quiver of B' as in the proof of [8, Theorem 3.7]. Then all paths in B' have fewer maximal paths of degree v_A, and $A/\mathcal{J}$ is the relatively free algebra of B'. □

We say that a T-ideal $\mathcal{I}$ of $F\{x\}$ is **representable** if $F\{x\}/\mathcal{I}$ is a representable algebra.

Theorem 5.6. *Suppose A is a relatively free, affine PI-algebra over a field F. Then any chain of T-ideals in the free algebra of $F\{x\}$ ascending from $\mathrm{id}(A)$ must terminate.*

Proof. First, we need to move to representable affine algebras. But, by Proposition 5.4, the T-ideal of A contains the T-ideal of a finite dimensional algebra, so we can replace A by that algebra.

We want to show that any ascending chain of T-ideals

$$\text{(22)} \qquad \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots$$

in the free algebra $F\{x\}$, with $\mathcal{I}_1 = \mathrm{id}(A)$, stabilizes. For each j, let $\mathcal{I}_j^{(0)} \subseteq \mathcal{I}_j$ denote the T-ideal of A generated by symmetrized $\bar{q}$-characteristic coefficient-absorbing polynomials of $\mathcal{I}_j$ having a nonzero specialization with maximal degree vector.

Then we get the chain

$$\text{(23)} \qquad \mathcal{I}_1^{(0)} \subseteq \mathcal{I}_2^{(0)} \subseteq \mathcal{I}_3^{(0)} \subseteq \cdots .$$

Let Γ be the full quiver of A, so $\mathrm{id}(A) = \mathrm{id}(\Gamma)$. Let v_A denote the maximal degree vector for a nonzero evaluation of some polynomial in A. Since $\mathcal{I}_j^{(0)}$ is Noetherian, by Theorem 4.28 applied to Theorem 4.26, we can define $\mathcal{I}_j^{(1)}$ to be the maximal T-ideal of $\hat{A}$ contained in $\mathcal{I}_j$.

Then, the chain

$$\text{(24)} \qquad \mathcal{I}_1^{(1)} \subseteq \mathcal{I}_2^{(1)} \subseteq \mathcal{I}_3^{(1)} \subseteq \cdots$$

of ideals is in the Noetherian algebra $\hat{A}$, and thus stabilizes at some $\mathcal{I}_{j_0}^{(1)}$.

Passing to $A/\mathcal{I}_{j_0}^{(1)}$, we may assume that $\mathcal{I}_j^{(1)} = 0$ for each $j > j_0$. Hence, $A/\mathcal{I}_{j_0}^{(1)} \subseteq \hat{A}/\mathcal{I}_{j_0}^{(1)}$ is representable. If $\mathcal{I}_j^{(0)}$ were nonzero, then the fully hiked polynomial of some $0 \neq f \in \mathcal{I}_j^{(0)}$ would be in $\mathcal{I}_j^{(1)} = 0$, a contradiction. Thus, $\mathcal{I}_j^{(0)} = 0$ for each $j > j_0$. In other words, $\mathcal{I}_j$ has only zero evaluations in degree v_A.

Finally, let $\mathcal{J}$ be the T-ideal defined in Lemma 5.5. Thus $\mathcal{J} \cap \mathcal{I}_j = \mathcal{I}_j^{(0)}$, so passing to $A/\mathcal{J}$, which is relatively free and representable by Lemma 5.5, we lower the maximum degree vector, and conclude by induction.

But these lift to a chain of ideals of the Noetherian algebra $\hat{A}$, which must stabilize, and thus (24) stabilizes at some $\mathcal{I}_{j_0}^{(0)}$. Passing to $A/\mathcal{I}_{j_0}^{(0)}$ and starting the chain at j_0, we may assume that $\mathcal{I}_j^{(0)} = 0$ for each j.

We let $v_{\mathcal{B}}$ be the degree vector of some A-admissible polynomial, and let $\mathcal{I}_{\mathcal{B}}$ be the T-ideal of Theorem 4.24, which shows that $\mathcal{I}_j \cap \mathcal{I}_{\mathcal{B}} = 0$ for each $j > j_0$. Let $\overline{\mathcal{I}_j} = (\mathcal{I}_j + \mathcal{I}_{\mathcal{B}})/\mathcal{I}_{\mathcal{B}}$, a T-ideal of the relatively free algebra $F\{x\}/\mathcal{I}_{\mathcal{B}}$ for each $j > j_0$. By induction on the maximal degree vector of the quiver, applied to the relatively free algebra $F\{x\}/\mathcal{I}_{\mathcal{B}}$, the chain of T-ideals

$$(25) \qquad \overline{\mathcal{I}_{j+1}} \subseteq \overline{\mathcal{I}_{j+2}} \subseteq \overline{\mathcal{I}_{j+3}} \subseteq \cdots$$

must stabilize, so we conclude that the original chain of T-ideals must stabilize. $\quad\square$

6. A SOLUTION OF SPECHT'S PROBLEM FOR PI-PROPER T-IDEALS OF AFFINE ALGEBRAS OVER ARBITRARY COMMUTATIVE NOETHERIAN RINGS

Using the same ideas, we can finally prove Specht's problem for affine PI-algebras over a commutative Noetherian ring C. Our strategy is to reduce to algebras over fields, since this case is already solved. The argument is based on a formal reduction from algebras over rings to algebras over fields, much of which we formulate rather generally for algebras which are not necessarily associative. Accordingly, we fix a given algebraic variety $\mathcal{V}$ of algebras, and consider Specht's problem for algebras in $\mathcal{V}$. When necessary, we take $\mathcal{V}$ to be the variety of associative algebras.

One can construct the free nonassociative algebra, whose elements are polynomials, where we also write them in parentheses to indicate the order of multiplication. The **T-ideal** of a set of polynomials **in an algebra** A is the ideal generated by all substitutions of these polynomials in A. The variety $\mathcal{V}$ itself is defined by a T-ideal $\mathcal{I}_{\mathcal{V}}$ of identities of the free nonassociative algebra, and the corresponding factor algebra is the relatively free algebra of the variety $\mathcal{V}$. (For example, when $\mathcal{V}$ is the variety of associative algebras, $\mathcal{I}_{\mathcal{V}}$ is generated by the **associator** $(x_1 x_2)x_3 - x_1(x_2 x_3)$.) Specht's problem now is whether every chain of ascending T-ideals containing the associator stabilizes or, equivalently, if every T-ideal is finitely based modulo the T-ideal $\mathcal{I}_{\mathrm{assoc}}$ of the associator.

There is an extra technical issue, since usually the definition of PI requires that the ideal of the base ring generated by the coefficients of the PIs is all of C. (This is obviously the case when C is a field.) We call such a T-ideal *PI-proper*, and start in this section with that case. Finally, in §7 we prove the general result for T-ideals which are not necessarily PI-proper.

6.1. **Reduction to algebras over integral domains.** The considerations of this section apply to arbitrary algebraic varieties (not necessarily associative).

Definition 6.1. We say that a Noetherian ring C is $\mathcal{V}$-**Specht** if Specht's problem has a positive solution in the variety $\mathcal{V}$ for PI-proper T-ideals defined over C, i.e., any PI-proper T-ideal generated by polynomials $f_1, f_2, \ldots$ is finitely based modulo $\mathcal{I}_{\mathcal{V}}$. We say that C is **almost $\mathcal{V}$-Specht** if C/I is $\mathcal{V}$-Specht for every nonzero ideal I of C. If $\mathcal{V}$ is not specified here, it is assumed to be the variety of associative algebras.

Remark 6.2. Here is the general reduction of Specht's problem to the case when C is an integral domain. We need to show that any T-ideal generated by polynomials $f_1, f_2, \ldots$ is finitely based modulo $\mathcal{I}_{\mathcal{V}}$. Let $\mathcal{I}_j$ be the T-ideal generated by $f_1, \ldots, f_j$. By Noetherian induction, we may assume that C is almost $\mathcal{V}$-Specht. Suppose $c_1 c_2 = 0$ for $0 \neq c_1, c_2 \in C$. By Noetherian induction, the system $\{f_j\}$ is finitely based over $c_2 A$, so there is some j_0 for which each $f_j = g_j + c_2 h_j$, where $g_j \in \mathcal{I}_{j_0}$

and h_j is arbitrary. The polynomials f_j can be replaced by $c_2 h_j$ for all $j > j_0$. But the T-ideal generated by $\{c_2 h_j : j > j_0\}$ in $A/c_1 A$ is finitely based, by Noetherian induction over $C/c_1 C$.

Thus, we are done unless $c_1 c_2 \neq 0$ for all $0 \neq c_1, c_2 \in C$, implying that C is an integral domain.

6.2. **Reduction to prime power torsion.** By Remark 6.2, we may assume from now on that C is an almost $\mathcal{V}$-Specht integral domain. Also, we assume that C is infinite, since otherwise C is a field, and we have solved the case of fields. To proceed further, we also introduce torsion in the opposite direction.

Definition 6.3. Let M be a module over a commutative integral domain C. For $z \in C$ define $\mathrm{Ann}_M(z) = \{a \in M : za = 0\}$.

$\mathrm{Ann}_M(z)$ is called the z-**torsion** of M. M is z-**torsionfree** if $\mathrm{Ann}_M(z) = 0$. For $I \triangleleft C$ and a C-module M, define $\mathrm{tor}_I(M) = \bigcup_{0 \neq z \in C} \mathrm{Ann}_M(z)$, a C-submodule of M since C is an integral domain.

The z-**torsion index** of M is k if the chain

$$\mathrm{Ann}_M(z) \subseteq \mathrm{Ann}_M(z^2) \subseteq \cdots \subseteq \mathrm{Ann}_M(z^k) \subseteq \cdots$$

stabilizes at k. Reversing Definition 5.1, we say that M is z'-**torsionfree** if its w-torsion submodule is 0 for each prime w not dividing z.

Lemma 6.4. *If A is a relatively free algebra and $c \in C$, then $\mathrm{Ann}_A c$ is a T-ideal, with $cA \cong A/\mathrm{Ann}_A c$ as C-modules (but not necessarily as C-algebras). We have an inclusion-reversing map $\{Ideals\ of\ C\} \to \{T\text{-}Ideals\ of\ A\}$ given by $I \mapsto \mathrm{tor}_I(A)$.*

Proof. $\mathrm{Ann}_A c$ is clearly a T-ideal, by Lemma 1.1, and the rest of the first assertion is standard. The second assertion is likewise clear. $\square$

In any commutative Noetherian domain, any element can be factored as a finite product of irreducible elements (although not necessarily uniquely).

Proposition 6.5. *Any module over an integral domain C whose torsion involves only finitely many irreducible elements is a subdirect product of finitely many z'-torsionfree modules, where z ranges over these irreducible elements of C.*

Proof. Follows at once from the lemma. $\square$

The following is a closely related result, which we record for reference in future work:

Proposition 6.6. *Any Noetherian $\mathbb{Z}$-module is a subdirect product of finitely many p'-torsionfree modules, where p ranges over the prime numbers.*

6.3. **Homogeneous T-ideals.** Recall that a polynomial is **homogeneous** if for each indeterminate x_i each of its monomials has the same degree in the indeterminate x_i. Since the degrees grade the free algebra, any polynomial has a unique decomposition as a sum of homogeneous polynomials, which we call its **homogeneous components**. Recall that a T-ideal $\mathcal{I}$ is **homogeneous** if it contains all of the homogeneous components of each of its polynomials.

Remark 6.7. Although any homogeneous T-ideal is clearly generated by homogeneous polynomials, in general, homogeneous polynomials need not generate a homogeneous T-ideal, because of the vagaries of the quasi-linearization procedure (see [9, Example 2.2] and [5, Exercise 13.10]).

Definition 6.8. A set of polynomials S is **ultra-homogeneous** if it contains the homogeneous component of every element in S, as well as of the quasi-linearizations of all polynomials in S.

The **ultra-homogeneous closure** $\overline{S}_{\text{uh}}$ of a set S is the intersection of all ultra-homogeneous sets containing it. (The ultra-homogeneous closure of a finite set is finite, since the procedure terminates for each polynomial after finitely many steps.)

The **homogeneous socle** $\underline{\mathcal{I}}_{\text{soc}}$ of a T-ideal $\mathcal{I}$ is the union of homogeneous T-ideals contained in $\mathcal{I}$.

Note that any homogeneous T-ideal $\mathcal{I}$ contains the homogeneous components of the quasi-linearizations of each of its polynomials, so is automatically ultra-homogeneous.

Proposition 6.9. *The T-ideal generated by an ultra-homogeneous set of polynomials S is homogeneous.*

Proof. We need to show that the homogeneous components of any substitution remain in the T-ideal $\mathcal{I}$ generated by $S = \overline{S}_{\text{uh}}$. By definition of quasi-linearization, it is enough to check this for monomial substitutions. But these are specializations of substitutions of letters (taking a different letter for each monomial), and thus are specializations of the homogeneous components of the quasi-linearizations, which by definition are in $\mathcal{I}$. $\square$

In particular every set of multilinear identities generate a homogenous T-ideal.

Corollary 6.10. *Let $\mathcal{V}$ be a variety satisfying the ACC on homogeneous T-ideals. Then every homogeneous T-ideal is finitely based.*

Proof. The ultrahomogeneous closure of a finite set of polynomials is finite. $\square$

Let $z \in C$ be any nonzero element. Our overall goal would be to prove formally that if every field is $\mathcal{V}$-Specht, then every commutative Noetherian ring is $\mathcal{V}$-Specht. Unfortunately, this is not quite in our grasp, since one detail still relies on associativity. We can prove the following theorem:

Theorem 6.11. *Let $\mathcal{V}$ be a variety of algebras such that every field is $\mathcal{V}$-Specht. If an integral domain C is almost $\mathcal{V}$-Specht, and is $\mathcal{V}$-Specht with respect to homogeneous T-ideals, then C is $\mathcal{V}$-Specht.*

Taking $\mathcal{V}$ to be the class of associative algebras, we conclude by proving that if a Noetherian ring C is almost Specht and every field is Specht, then C is Specht with respect to homogeneous T-ideals.

This will affirm Specht's problem for affine PI-algebras over an arbitrary Noetherian ring, and together with Theorem 7.6 below will affirm Specht's problem for arbitrary affine algebras over a Noetherian ring.

We deal with the reduction for other varieties in a subsequent paper.

6.3.1. *Proof of Theorem* 6.11. Although we are working in the context of associative algebras, the proof of Theorem 6.11 also works analogously for nonassociative algebras.

Lemma 6.12. *Suppose $\mathcal{I}$ is a T-ideal, and $f = \sum f_i \in \mathcal{I}$ has total degree n (where f_i are the homogeneous components). Then for every Vandermonde determinant d of order n, $d\overline{\langle f \rangle}_{\text{uh}} \subseteq \mathcal{I}$.*

Proof. Substitute $\lambda_j x_j$ for x_j, for fixed j. Let d denote the determinant of the Vandermonde matrix (λ_j^i) and $i = 1, \ldots, n$. The homogeneous components of df are in $\mathcal{I}$, by the usual Vandermonde argument of multiplying by the adjoint matrix. □

Lemma 6.13. *Suppose C is $\mathcal{V}$-Specht with respect to homogeneous T-ideals, and $\mathcal{I}$ is a proper T-ideal that properly contains its homogeneous socle $\mathcal{I}_0$. Then $\mathcal{I}$ contains a homogeneous T-ideal of the form $d\mathcal{I}_1$, where $\mathcal{I}_1 \supset \mathcal{I}_0$ is a finitely based, proper T-ideal. (Here d is a product of Vandermonde determinants.)*

Proof. Take a proper polynomial $f \in \mathcal{I} \setminus \mathcal{I}_0$. By Lemma 6.12 there is $0 \neq d_1 \in C$ such that $d_1 f_i \in \mathcal{I}$, where f_i are the homogeneous components of the quasi-linearizations of f. Continuing with the quasi-linearization procedure, which is finite, we see by induction that there is some d' such that $d' g_{i,j} \in \mathcal{I}$, for each component $g_{i,j}$ in the various quasi-linearizations, implying $d' d_1 \overline{\langle f \rangle}_{\mathrm{uh}} \subseteq \mathcal{I}$, as desired. □

Note that we had to take the ultra-homogeneous closure of a polynomial, and not a T-ideal, to remain with a finite set of polynomials. We are ready to prove Theorem 6.11.

Proof. Let A be an algebra in $\mathcal{V}$. We introduce some notation: Given $z \in C$ and a T-ideal Γ, we write $\mathcal{I}_\Gamma(z)$ for the kernel of the composite map $A \to zA \to zA/(\Gamma \cap zA)$. By hypothesis we can take $z \in C$ with $\overline{\mathcal{I}_\gamma(z)}_{\mathrm{uh}}$ maximal, and we write z_Γ for z and $\mathcal{J}(\Gamma)$ for $\mathcal{I}_\gamma(z)$. $\mathcal{J}(\Gamma) = \underline{\mathcal{J}(\Gamma)}_{\mathrm{soc}}$, since otherwise we could use Lemma 6.13 to increase $\underline{\mathcal{J}(\Gamma)}_{\mathrm{soc}}$, contrary to its definition. Hence, $\mathcal{J}(\Gamma)$ is already homogeneous.

Given a chain $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ of T-ideals of A, we see by the hypothesis on homogeneous T-ideals that there is i such that $\mathcal{J}(\Gamma_j) = \mathcal{J}(\Gamma_i)$ for all $j \geq i$. Write $\hat{z} = \prod_{j \leq i} z_{\Gamma_j}$. Then the $\mathcal{J}(\Gamma_j) \cap \hat{z} A$ also stabilize. If some $\Gamma_j \cap \hat{z} A$ properly contains $\mathcal{J}(\Gamma_i) \cap \hat{z} A$, then it has a (nonhomogeneous) polynomial f and thus contains $\hat{z} \overline{f}_{\mathrm{uh}}$, which is impossible unless $\hat{z} C$ is a proper ideal of C.

But $\mathcal{I}_j(A)/\hat{z}\mathcal{I}_j(A)$ is a T-ideal over $C/\hat{z}C$, so we conclude by Noetherian induction. □

6.4. Conclusion of the solution of Specht's problem for arbitary affine PI-algebras over Noetherian rings.
We start with some general considerations that can be used for arbitrary varieties. The following well-known fact is a key ingredient, yielding a tool for applying Noetherian induction.

Lemma 6.14 (Baby Fitting Lemma). *Let M be a C-module, with $z \in C$, and take any $k \in \mathbb{N}$. Suppose $\mathrm{Ann}_M(z^{k+1}) \subseteq \mathrm{Ann}_M(z^k)$. Then $z^k M \cap \mathrm{Ann}_M(z) = 0$.*

Proof. If $z^k a \in \mathrm{Ann}_M(z)$, then $z^{k+1} a = 0$, implying $z^k a = 0$ by assumption. □

We also need some easy facts from module theory.

Lemma 6.15. *Let M, N be modules over a commutative ring C. Let $f : M \to N$ be a homomorphism of modules.*

(i) For every $z, z' \in C$, if the induced homomorphisms $f' : M/z'M \to N/z'N$ and $f'' : z'M/zz'M \to z'N/zz'N$ are 1:1, then so is the induced homomorphism

$$f''' : M/zz'M \to N/zz'N.$$

(ii) If the induced homomorphisms $z^i M/z^{i+1}M \to z^i N/z^{i+1}N$ are $1:1$ for every $0 \le i < k$, then the induced homomorphism $M/z^k M \to N/z^k N$ is $1:1$ as well.

Proof. (i) If $a \in \ker f'''$, then $a + z'M \in \ker f' = 0$, implying $a \in \ker f'' = 0$.

(ii) By induction on k, taking $z' = z^{k-1}$ in the previous lemma. $\qquad\square$

Let S denote the monoid generated in C by z. Recall that
$$S^{-1}M = \left\{ s^{-1}a : \; s \in S, a \in M \right\},$$
where $s^{-1}a = s'^{-1}a'$ if there is $s_0 \in S$ such that $s_0(s'a - sa') = 0$. In particular $s^{-1}a = 0$ if there is $s_0 \in S$ such that $s_0 a = 0$.

Lemma 6.16. *Let $f : M \to N$ be a homomorphism of C-modules, and let $z \in C$. Let $f' : S^{-1}M \to S^{-1}N$ and $f_i : M/z^i M \to N/z^i N$ be the induced homomorphisms, where S is the monoid generated by z.*

   (1) *Assume N has z-torsion index k. If every f_i is one-to-one and f' is onto, then the restriction $f|_{z^k M} : z^k M \to z^k N$ is onto.*

   (2) *Assume M has z-torsion index k. If f' is one-to-one, then the restriction $f|_{z^k M} : z^k M \to z^k N$ is one-to-one.*

   (3) *Assume M has z-torsion index k. If f_k and f' are one-to-one, then f is one-to-one.*

Proof.   (1) Let $b \in N$. By assumption there is an element $z^{-\ell}a \in S^{-1}M$ such that $z^{-\ell}f(a) = f'(z^{-\ell}a) = 1^{-1}b \in S^{-1}N$, so for some $\ell' \ge 0$ we have that $z^{\ell'}f(a) = z^{\ell+\ell'}b$. Then $f_{\ell+\ell'}(z^{\ell'}a + z^{\ell+\ell'}M) = f(z^{\ell'}a) + z^{\ell+\ell'}N = 0$, so $z^{\ell'}a = z^{\ell+\ell'}a'$ for some $a' \in M$. But now $z^{\ell+\ell'}b = f(z^{\ell'}a) = z^{\ell+\ell'}f(a')$, so $z^{\ell+\ell'}(b - f(a')) = 0$. Since the torsion index of N is k, we have that $z^k b = f(z^k a')$.

   (2) Let $a \in M$ be such that $f(z^k a) = 0$. Then $f'(1^{-1}z^k a) = 1^{-1}z^k f(a) = 0$ in $S^{-1}N$, so by assumption $1^{-1}z^k a = 0$, namely for some $\ell \ge 0$, $z^{k+\ell}a = 0$. Since k is the z-torsion index of M, we have that $z^k a = 0$.

   (3) Let $a \in M$ be such that $f(a) = 0$. Then $f_k(a + z^k M) = 0 + z^k N$, so $a \in z^k M$, but then (2) implies that $a = 0$. $\qquad\square$

Finite torsion index is essential in Lemma 6.16 (which is why Lemma 6.20 below only applies to homogeneous T-ideals).

Example 6.17. (An example where the restrictions f' and f_i are isomorphisms, but $f|_{z^k M} : z^k M \to z^k N$ is neither onto nor one-to-one). Let $C = F[z]$, and $P = F[[z^{-1}]]/F$ with the natural C-module structure. Since multiplication by z is onto, $P/z^i P = 0$ for every i, and $S^{-1}P = 0$ since $1^{-1}(z^{-i}) = z^{-i}z^i(z^{-i}) = z^{-i}0 = 1^{-1}0$. Let f be the zero map from $P \oplus 0$ to $0 \oplus P$; it is neither one-to-one nor onto, but the induced maps f_i and f' are clearly (trivial) isomorphisms. Indeed, P has infinite z-torsion index.

Remark 6.18. For any $z \in C$, $zA \cong A/\mathrm{Ann}\, z$ is a T-ideal of A.

To progress with the proof over an arbitrary base ring, we first need the special case where the T-ideal contains a representable T-ideal.

Theorem 6.19 (Small Specht Theorem). *Let C be an almost Specht, commutative Noetherian ring, and A an affine PI-algebra containing a representable T-ideal $\mathcal{I}$, i.e., the algebra $A/\mathcal{I}$ is representable. Then any chain of T-ideals in the free algebra of $C\{x\}$ ascending from $\mathrm{id}(A)$ stabilizes.*

Proof. By Remark 6.2, C is an integral domain. We need to show that any ascending chain of PI-proper T-ideals

$$(26) \qquad \mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots$$

of A, stabilizes. Since $\mathcal{I} \subseteq \mathcal{I}_1$, we may replace A by $A/\mathcal{I}$, and assume that A is representable. We view $A \subseteq M_n(K)$, where K is an algebraically closed field containing C. If C is finite, then it is a field, and we are done by Theorem 5.6. So we may assume that C is an infinite integral domain. Denoting AK as A_K, we work with respect to a quiver Γ of A_K as a K-algebra.

As in Theorem 5.6, let $\mathcal{I}_j^{(1)}$ be the maximal subideal of $\mathcal{I}_j$ closed under multiplication by $\hat{C}$ of Theorem 4.26. Thus

$$(27) \qquad \mathcal{I}_1^{(1)} \subseteq \mathcal{I}_2^{(1)} \subseteq \mathcal{I}_3^{(1)} \subseteq \cdots$$

are ideals in the Noetherian algebra $\hat{A} = \hat{C}A$, so this chain stabilizes, and we may assume $\mathcal{I}_j^{(1)} = \mathcal{I}_{j_0}^{(1)}$ for $j > j_0$.

For a T-ideal $\mathcal{I}$ of A, let $\overline{\mathcal{I}} = K\mathcal{I}$, taken in A_K. Define $\widetilde{\mathcal{I}} = K\mathcal{I} \cap A \supseteq \mathcal{I}$. Let $A' = A/\mathcal{I}_{j_0}^{(1)}$. Passing down to A', we shall pass further to $A/\widetilde{\mathcal{I}_{j_0}^{(1)}}$.

The quotient $\widetilde{\mathcal{I}_{j_0}^{(1)}}/\mathcal{I}_{j_0}^{(1)}$ is torsion, so there is $0 \neq z \in \hat{C}$ such that $z\widetilde{\mathcal{I}_j^{(1)}} = z\widetilde{\mathcal{I}_{j_0}^{(1)}} \subseteq \mathcal{I}_{j_0}^{(1)}$. The chain $\operatorname{Ann}_{\hat{A}} z \subseteq \operatorname{Ann}_{\hat{A}} z^2 \subseteq \cdots \subseteq \operatorname{Ann}_{\hat{A}} z^k \subseteq \cdots$ stabilizes at some k, by the Noetherianity of $\hat{A}$. Now, applying the baby Artin-Rees lemma to $\hat{A}/\mathcal{I}_{j_0}^{(1)}$, we see that

$$z^k \hat{A} \cap \widetilde{\mathcal{I}_j^{(1)}} \subseteq \mathcal{I}_{j_0}^{(1)}.$$

In particular the natural map

$$A' \to (A'/z^k A') \oplus (A/\widetilde{\mathcal{I}_{j_0}^{(1)}})$$

is an injection. The image of the chain (26) of the first summand on the right stabilizes by applying Noetherian induction. Thus, we pass to the second summand of the right, which has no C-torsion. Letting $\mathcal{J}$ be the ideal constructed in Lemma 5.5, we have for every $j > j_0$ that $\mathcal{I}_j \cap \mathcal{J} = 0$ in $A_K/A_K\mathcal{I}_{j_0}^{(1)}$ as in the last paragraph of the proof of Theorem 5.6. Hence, a fortiori, $\mathcal{I}_j \cap \mathcal{J} = 0$ in $A/\widetilde{\mathcal{I}_{j_0}}$. We are done by induction on the degree vector. $\qquad \square$

Lemma 6.20. *Suppose $z \in C$ such that C/zC and $C[z^{-1}]$ are $\mathcal{V}$-Specht. Then C satisfies the ACC on homogeneous T-ideals from $\mathcal{V}$.*

Proof. By induction on the length of z as a product of primes, we may assume that z is prime. Let A be an affine algebra over C, and let

$$\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots$$

be an ascending chain of T-ideals in A.

Let $A_i = A/\mathcal{I}_i$, and consider the infinite commutative diagram

$$(28) \qquad \begin{array}{ccccccccc}
A_1/zA_1 & \longrightarrow & zA_1/z^2A_1 & \longrightarrow & z^2A_1/z^3A_1 & \longrightarrow & z^3A_1/z^4A_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_2/zA_2 & \longrightarrow & zA_2/z^2A_2 & \longrightarrow & z^2A_2/z^3A_2 & \longrightarrow & z^3A_2/z^4A_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_3/zA_3 & \longrightarrow & zA_3/z^2A_3 & \longrightarrow & z^2A_3/z^3A_3 & \longrightarrow & z^3A_3/z^4A_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where the left-to-right maps are multiplication by z, and the top-to-bottom arrows are the natural projections. So all the maps are projections.

We claim that outside a certain rectangle, all the maps in this infinite matrix are one-to-one. Indeed, the entries are algebras over C/zC, so each row stabilizes by assumption. Letting B_i denote the final algebra in row i, we obtain a chain of projections $B_1 \to B_2 \to \cdots$ which must also stabilize, proving that for some k_0, all of the rows stabilize after k_0 steps. We are done since each of the first k_0 columns stabilizes. It follows that when i is large enough, all the maps $z^j A_i/z^{j+1} A_i \to z^j A_{i+1}/z^{j+1} A_{i+1}$ are isomorphisms, so by Lemma 6.15.(ii), the natural projection $A_i \to A_{i+1}$ induces isomorphisms $A_i/z^j A_i \to A_{i+1}/z^j A_{i+1}$ for every j.

Similarly, the chain of projections

$$C[z^{-1}]A_1 \to C[z^{-1}]A_2 \to \cdots$$

stabilizes by the assumption on $C[z^{-1}]$.

Let i be large enough. Since $\mathcal{I}_i \subseteq \mathcal{I}_{i+1}$ are homogeneous, the natural projection $A_i \to A_{i+1}$ preserves the degree grading. Each homogeneous component is finite as a C-module since A is affine, and thus Noetherian, and therefore has finite z-torsion index. By Lemma 6.16.(3), the map in each component is one-to-one, proving that $\mathcal{I}_i = \mathcal{I}_{i+1}$. $\square$

The main idea in the proof given above is a simple version of a spectral sequence. Having proved a special case, we do a more general case (in fact, our most general version holds for an arbitrary variety $\mathcal{V}$ of algebras).

Theorem 6.21. *Suppose the relatively free algebra A with respect to a T-ideal $\mathcal{I}$ is z'-torsionfree for some $z \in C$, where $\mathcal{I}$ is generated by a polynomial all of whose coefficients are ± 1. Then any increasing chain of T-ideals of A starting with $\mathcal{I}$ must terminate, for any commutative Noetherian ring C.*

Proof. Writing z as a product of primes, we may assume that z is prime. Let C_0 be the subring of C generated by 1. Letting L be the field of fractions of $C_0/(C_0 \cap zC)$, we pass to $A \otimes_{C_0} L$, so we may assume that C_0 is a field. (If there is no p-torsion at any step, then we can localize at the natural numbers and reduce to the case of $\mathbb{Q}$-algebras, which was solved by Kemer.) But $C_0[z]$ thus is a PID, and by the argument in Lemma 6.20, we can break up our chain into chains of T-ideals defined over C/zC, so we conclude by Lemma 6.20. $\square$

So far, these arguments have been applied to arbitrary varieties, and in fact there are Lie, alternative and Jordan versions of Iltyakov [14, 15] and Vais and

Zelmanov [27]; their proofs are rather delicate, in part because it is still unknown whether any alternative, Lie, or Jordan algebra satisfying a Capelli system of identities must satisfy all the identities of a finite dimensional algebra. Belov [3] obtained a version of the Small Specht Theorem (Theorem 6.19) for classes of algebras of characteristic 0 asymptotically close to associative algebras; this includes alternative and Jordan algebras.

Our method here is to develop some theory to take care of torsion in polynomials, to conclude the proof of Theorem 6.22 below. In other words, we need some local-global correspondence that will enable us to pass from the global situation with torsion to the local situation without torsion. Our main tool is Proposition 6.5, but this only enables us to handle a finite number of irreducible elements of C producing torsion, whereas there might be an infinite number of such elements. Thus we need some way of cutting down from infinite to finite.

The most direct argument relies on an (associative) result only available in Russian. Procesi asked whether the kernel of the canonical homomorphism $\mathrm{id}(\mathrm{M}_d(\mathbb{Z})) \to \mathrm{id}(\mathrm{M}_d(\mathbb{Z}/p\mathbb{Z}))$ is equal to $p\,\mathrm{id}(\mathrm{M}_d(\mathbb{Z}))$.

Schelter and later Kemer [20] provided counterexamples, but Samoĭlov [25] showed that if $p > 2d$, the kernel of the canonical homomorphism $\mathrm{id}(\mathrm{M}_d(\mathbb{Z})) \to \mathrm{id}(\mathrm{M}_d(\mathbb{Z}/p\mathbb{Z}))$ is indeed equal to $p\,\mathrm{id}(\mathrm{M}_d(\mathbb{Z}))$. Unfortunately, this result appears so far only in his doctoral dissertation [25].

Thus, we give two versions for the conclusion of the proof of the next theorem, the first relying on Samoĭlov's Theorem, and the second for those readers who would prefer a full proof of Theorem 6.22 in English. Another advantage of the second proof is that its reduction argument works for arbitrary varieties.

Theorem 6.22. *Any PI-proper T-ideal $\mathcal{I}$ of $C\{x_1, \ldots, x_\ell\}$ is finitely based, for any commutative Noetherian ring C.*

Proof. Let A be the relatively free algebra of $\mathcal{I}$. We can replace $\mathcal{I}$ by the T-ideal of a PI-proper polynomial f contained in it. But by Amitsur [5, Theorem 3.38], any PI-algebra satisfies a power of a standard polynomial, so we may assume f is such a polynomial, and thus has all nonzero coefficients in $\{\pm 1\}$.

Consider the localization $A \mapsto A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, viewed as a C_p/pC_p-algebra. By a theorem of Bergman and Dicks [10], there is a canonical homomorphism of A to a representable algebra, whose kernel M, in view of Lewin's theorem, vanishes modulo p for any prime p, i.e., when we map $A \mapsto A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, viewed as a C_p/pC_p-algebra. But the map $A \to A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is faithful whenever the kernel of the canonical homomorphism $\mathrm{id}(\mathrm{M}_d(\mathbb{Z})) \to \mathrm{id}(\mathrm{M}_d(\mathbb{Z}/p\mathbb{Z}))$ is equal to $p\,\mathrm{id}(\mathrm{M}_d(\mathbb{Z}))$, where d is the size of matrices in the representation, which by Samoĭlov's Theorem [25] happens when $p > 2n$, showing that A is $(2n)!$-torsionfree. The claim then follows by Theorem 6.21. $\qquad\square$

We now turn to the second proof.

Remark 6.23. The point here is that in view of Proposition 2.30, taking M to be the T-ideal generated by f as notated there, assuming that A satisfies a Capelli identity c_{n+1} and we are in a matrix component of degree n, the relatively free algebra A is integral over the affine C-algebra $C[\xi]$ where ξ denotes the set of characteristic coefficients formally corresponding to the finitely many δ operators. Unfortunately, this case can be assured only when C is a field, but by a careful use of localization we can formulate a local-global framework in which we can utilize this situation.

Second proof of Theorem 6.22. Let $A = C\{a_1, \ldots, a_\ell\}$ be the relatively free algebra of $\mathcal{I}$. We formulate an inductive argument in analogy to Theorem 5.6. In order to apply the theory of full quivers, we need to pass to some field. By Remark 6.2 we may assume that C is an integral domain. Let F denote the field of fractions of C, and let $A_F := A \otimes_C F$. We consider $\mathcal{I} \otimes F$ which contains the ideal of identities of A_F. Unfortunately, $(\mathcal{I} \otimes F) \cap C\{x\}$ might properly contain $\mathcal{I}$. If the torsion over C only involved finitely many primes we could handle this by means of Proposition 6.5, but this need not be the case. Thus, we need a more delicate argument which enables us to relate $\mathcal{I}$ with $\mathcal{I}_F$.

Step 1. We start with a proper PI of A. As mentioned in the first proof, Amitsur [5, Theorem 3.38] says that every PI-algebra satisfies some power of a standard identity, which we denote here as f. Let $\mathcal{I}_0$ denote the T-ideal of $C\{x\}$ generated by f, contained in $\mathcal{I}$, so $\mathcal{I}_0 \otimes F$ is the T-ideal of $F\{x\}$ generated by f, contained in $\mathcal{I} \otimes F$. The relatively free algebra $F\{x\}/(\mathcal{I}_0 \otimes F)$ has some full quiver Γ_1. Although Γ_1 does not have much to do with the original algebra A, it provides a base for an inductive argument, as well as a handle for using our field-theoretic results. Since the chain of reductions of any full quiver must terminate after a finite number of steps, we induct on Γ_1.

We need to show that every chain $\mathcal{C} = \{\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_3 \subseteq \cdots\}$ of T-ideals ascending from $\mathcal{I}_0$ stabilizes. Over the field F, we could do this by the argument of Theorem 5.6, which we recall is achieved by hiking f, obtaining matrix characteristic coefficients for the evaluations of a maximal branch of Γ_1, redefining these in terms of elements of the T-ideal $\mathcal{I}_0 \otimes F$, using Theorem 4.28 to show that this part of the T-ideal is Noetherian, and then modding it out and applying Noetherian induction. Unfortunately, working over C might involve C-torsion, which could collapse infinite chains when passing to the field of fractions, F. We can use Proposition 6.5 to eliminate torsion involved with a given finite set of elements of C, so our strategy is to show how the whole process just described can be achieved over a localization of C by a finite number of elements, which are found independently of the specific chain $\mathcal{C}$. Thus, we can work over this localization just as well as over F, and pass back to C by means of Proposition 6.5.

Step 2. We rely heavily on Proposition 6.5 in order to eliminate torsion involved with a given finite set of elements of C, with the aim of modifying A in order to make it more compatible with Γ_1. We say that a T-ideal of $F\{x\}$ is C-**expanded** if it is generated by polynomials $\subset C\{x\}$. We extend $f_1 = f$ to a set $\{f_1, \ldots, f_k\} \subset C\{x\}$ generating a maximal possible C-expanded T-ideal of $F\{x\}$ contained in $\mathcal{I} \otimes F$. (Such a finite set exists since we already have solved Specht's problem over fields, implying the $F\{x\}$ satisfies the ACC on C-expanded T-ideals.)

The coefficients of $f_1, \ldots, f_k$ involve only finitely many elements of C. Utilizing Proposition 6.5, we can localize at these primes to obtain a new base ring C' and assume that A has no torsion at the coefficients of the polynomials $f_1, \ldots, f_k$. Let $\mathcal{I}'$ (resp. $\mathcal{I}' \otimes F$) denote the T-ideal of $C'\{x\}$ (resp. of $F\{x\}$) generated by $f_1, \ldots, f_k$, whose full quiver over F is denoted as Γ_2. This might increase our C'-expanded T-ideal over F, requiring us to adjoin more polynomials, and thereby forcing us to localize by finitely many more primes, but the process must stop since $F\{x\}$ satisfies the ACC on C-expanded T-ideals. This achieves our goal of matching a T-ideal over C with a T-ideal over F.

Step 3. Our next goal is to hike to a $\bar{q}$-characteristic coefficient-absorbing polynomial. As in Lemma 5.5, we take a maximal path in the full quiver Γ_2. Its polynomial can be hiked to a finite set of polynomials $\tilde{f}_1, \ldots, \tilde{f}_m$. Unfortunately these might involve torsion with new primes of C'. But the torsion over C' in localizing these finitely many polynomials involves only finitely many prime elements in C', and by localizing at them we obtain a new base Noetherian ring C'' and an algebra $A'' = A \otimes C''$ over it. Now we can appeal again to Proposition 6.5 and replace A by A''; thereby, we may assume that A'' is z-torsionfree for the finitely many primes z at which we localized. (Perhaps $F\{x\}$ has more C''-expanded T-ideals than C'-expanded T-ideals, so we must return to Step 2 and then Step 3, but this loop must terminate since $F\{x\}$ satisfies the ACC on T-ideals.)

In this way, we avoid all torsion in computing the $\bar{q}$-characteristic coefficients in the maximal matrix components, and thereby perform these calculations in A''. In other words, we can use $\tilde{f}_1, \ldots, \tilde{f}_m$ (taken over C'') to calculate $\bar{q}$-characteristic coefficients of the products of the generators of A in terms of polynomials.

Starting with C'' we let $\mathcal{I}''$ (resp. $\mathcal{I}'' \otimes F$) be the T-ideal of $C''\{x\}$ (resp. of $F\{x\}$) generated by $f_1, \ldots, f_k$ and $\tilde{f}_1, \ldots, \tilde{f}_m$.

Step 4. This is the most delicate part of the proof. Our strategy in this case is to go back to mimic the proof of the field-theoretic case (Theorem 5.6), removing C-torsion step by step when we pass back from $F\{x\}$ to $C''\{x\}$. But we must be careful to do everything in a finite number of steps. We would like to appeal to compactness from logic, but the argument is more subtle, since certain steps cannot be put in quantitative form. In particular, we note that the chain

$$\{\mathcal{I}_j \otimes_C F : j \in \mathbb{N}\}$$

stabilizes at $\mathcal{I}_{j_0} \otimes_C F$ for some j_0, which we take to be $j_0 = 1$, and we define $\mathcal{I}^{(0)}_{1;F} \subseteq \mathcal{I}_1 \otimes_C F$ to be the T-ideal of A_F generated by symmetrized $\bar{q}$-characteristic coefficient-absorbing polynomials of $\mathcal{I}_1 \otimes_C F$ having a nonzero specialization with maximal degree vector, as described in the proof of Theorem 5.6.

This is generated by finitely many polynomials of A_F, which can be taken from A'' and define a T-ideal of A'' which we call $\mathcal{I}^{(0)}_1$. Working with $\mathcal{I}^{(0)}_{1;F}$ enables us to define finitely many characteristic coefficients which we define in terms of polynomials which we now call $g_1, \ldots, g_m \in C''\{x\}$. Inverting the torsion, i.e., localizing at some $z \in C''$, we now may assume that the g_i are C''-torsionfree (and nonzero since they localize to nonzero elements of $F\{x\}$).

We would like to use $g_1, \ldots, g_m$ to define "characteristic coefficients" for the elements of $\mathcal{I}^{(0)}_1 \subset A''$, but unfortunately these are no longer central. But inverting the C''-torsion of the $g_i g_j - g_j g_i$, $1 \leq i, j \leq m$, we may assume that $g_1, \ldots, g_m$ all commute, and $\mathcal{I}^{(0)}_1$ is a module over the commutative Noetherian ring $\hat{C} := C''[g_1, \ldots, g_m]$. This is enough for us to apply Theorem 4.28 to show that $\mathcal{I}^{(0)}_1$ is a finite module, and thus Noetherian. (We can define the δ-operators via Remark 6.23, together with the module M which is the T-ideal generated by $\tilde{f}$. Note that since we only need consider monomials up to a certain length, we need to adjoin only finitely many characteristic coefficients, again via localization and Proposition 6.5.)

In this way, after localizing by finitely many elements of C, we pass to finite modules over Noetherian rings.

After all of these localizations we have a new affine base ring $C''' \supset C''$, and we work over $\hat{C}''' := C'''[g_1, \ldots, g_m]$. We let $\mathcal{I}'''$ be the T-ideal of $C'''\{x\}$ generated by the new polynomials involved with these extra steps. Thus $\mathcal{I}''' \otimes F$ (resp. $\mathcal{I}''' \otimes \hat{C}$) is the T-ideal of $F\{x\}$ (resp. of $\hat{C}'''\{x\}$) generated by the same new polynomials.

If $\mathcal{I}' \otimes F \subset \mathcal{I}''' \otimes F$, then the quiver of the relatively free algebra $F\{x\}/(\mathcal{I}''' \otimes F)$ is a reduction of Γ_1, so we conclude by induction on the complexity of the quiver, in view of Lemma 4.30.

Thus, we may assume that

$$\mathcal{I}''' \otimes F = \mathcal{I}' \otimes F.$$

Next we look at $(\mathcal{I}''' \otimes \hat{C})/(\mathcal{I}' \otimes \hat{C})$. By assumption, this is a torsion submodule of the Noetherian module $\mathcal{I}_{1;F}^{(0)}/(\mathcal{I}' \otimes \hat{C})$ and thus is finite, and so if nonzero it is annihilated by some nonzero element $z \in C$. We can remove the z-torsion one final time (again via localization and Proposition 6.5), passing to a new base ring $C'''' \supset C'''$ and T-ideal $\mathcal{I}''''$ (resp. $\mathcal{I}'''' \otimes F$) of $C''''\{x\}$ (resp. of $F\{x\}$). If $\mathcal{I}' \otimes F \subset \mathcal{I}'''' \otimes F$, then the quiver of the relatively free algebra $F\{x\}/(\mathcal{I}'''' \otimes F)$ is a reduction of Γ_1, so we conclude by induction on the complexity of the quiver, in view of Lemma 4.30. Thus we may assume that $\mathcal{I}'''' \otimes F = \mathcal{I}' \otimes F$, and since z has been inverted in the localization we conclude that our ascending chain of T-ideals from $\mathcal{I}'$ lifts to an ascending chain of T-ideals from $\mathcal{I}''''$ and we are done by the process given in the proof of Theorem 5.6. (The point is that the argument of modding out a certain Noetherian submodule of each T-ideal in $A \otimes_C C''''$ is algorithmic, depending on computations involving a finite number of polynomials whose C-torsion we have removed.) This concludes the second proof of Theorem 6.22.

In summary, we have performed various procedures in order to enable us to reduce the quiver. These procedures involve a T-ideal of $F\{x\}$ which might increase because of the procedure, but must eventually terminate because $F\{x\}$ satisfies the ACC on T-ideals. But at this stage Step 3 does not vitiate Step 2, and we can conclude the proof using Step 4 to carve out representable T-ideals over C''''.

Alternatively, one could conclude by applying the compactness in logic to the proof of Kemer's theorem and checking that we only need finitely many elements, which can be computed. Fuller details of the compactness argument are forthcoming when we consider representability and the universal algebra version of Theorem 6.22.

7. The case where the T-ideals are not necessarily PI-proper

Using the same ideas, we can extend Theorem 6.22 still further, considering the general case where the T-ideals are not PI-proper; in other words, the ideals of C generated by the coefficients of the polynomials in the T-ideals of $C\{X\}$ do not contain the element 1. Towards this end, given a set S of polynomials in $C\{X\}$, define its **coefficient ideal** to be the ideal of C generated by the coefficients of the polynomials in S. We need a few observations about the multilinearization procedure.

Lemma 7.1. *If a T-ideal $\mathcal{I}$ contains a polynomial f with coefficient c, then $\mathcal{I}$ also contains a multilinear polynomial with coefficient c.*

Proof. First we note that one of the blended components of f has coefficient c, and then we multilinearize it. □

Lemma 7.2. *If c is in the coefficient ideal of a T-ideal $\mathcal{I}$ of $C\{x\}$, then some multilinear $f \in \mathcal{I}$ has coefficient c.*

Proof. If $c = \sum c_i s_i$ where s_i appears as a coefficient of $f_i \in \mathcal{I}$, then taking the $f_i = f_i(x_1, \ldots x_{m_i})$ to be multilinear, we may assume that c_i is the coefficient of $x_1 \cdots x_{m_i}$. Taking $m > \max\{m_i\}$, we see that the coefficient of $x_1 \cdots x_m$ in

$$\sum_i s_i f_i(x_1, \ldots x_{m_i}) x_{m_i+1} \cdots x_m$$

is 1.   □

Corollary 7.3. *A T-ideal is PI-proper iff its coefficient ideal contains 1.*

Proposition 7.4. *Suppose C is a Noetherian integral domain, and $\mathcal{I}$ is a T-ideal with coefficient ideal I. Then there is a polynomial $f \in \mathbb{Z}\{x\}$ for which $cf \in \mathcal{I}$ for all $c \in I$.*

Proof. Since C is Noetherian, we can write $I = \sum_{i=1}^{t} Cc_i$, and then it is enough to prove the assertion for $c = c_i$, $1 \le i \le t$.

We take the relatively free, countably generated algebra A whose generators $\{y_1, y_2, \ldots, \}$ are given the lexicographic order, and let M_m denote the space of multilinear words of degree m in $\{y_1, \ldots, y_m\}$. In view of Shirshov's Height Theorem [5, Theorem 2.3], the space $\sum_i c_i M_m$ has bounded rank as a $\mathbb{Z}$-module. On the other hand, there is a well-known action of the symmetric group S_m acting on the indices of $y_1, \ldots, y_m$ described in [5, Chapter 5]. In particular, [5, Theorem 5.51] gives us a rectangle such that any multilinear polynomial f whose Young diagram contains this rectangle satisfies $c_i f \in \mathcal{I}$.   □

Write $\mathcal{M}(K)$ for the generic $n \times n$ matrix algebra with characteristic coefficients over a commutative ring K. Zubkov [29] proved that the canonical map $\mathcal{M}(\mathbb{Z}) \to \mathcal{M}(\mathbb{Z}/p\mathbb{Z})$ has kernel $p\mathcal{M}(\mathbb{Z})$. (As noted above, this is false if one does not adjoin characteristic coefficients.) He also proved that the Hilbert series of the algebra $A \otimes_{\mathbb{Z}} \mathbb{Q}$ over $\mathbb{Q}$ and of $A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}/p\mathbb{Z}$ coincide, implying $\mathcal{I}$ is a free $\mathbb{Z}$-module.

Corollary 7.5. *If $\mathcal{I}$ is a T-ideal with coefficient ideal I, there is a PI-proper T-ideal of $C\{x\}$ whose intersection with $I\{x\}$ is contained in $\mathcal{I}$.*

Proof. We need to show that

$$(29) \qquad\qquad\qquad\qquad \mathcal{I} \cap cA = c\mathcal{I}$$

for any $c \in C$. We take the polynomial f of Proposition 7.4. In view of Proposition 5.4, the T-ideal of f contains the set of identities of some finite dimensional algebra, and thus of $M_n(C)$ for some n. Adjoining characteristic coefficients, we may replace $\mathcal{I}$ by a T-ideal of the free algebra with characteristic coefficients, and conclude with Zubkov's results [29] quoted above.   □

Theorem 7.6. *Any T-ideal in the free algebra $C\{x\}$ is finitely based, for any commutative Noetherian ring C.*

Proof. By Noetherian induction, we may assume that the theorem holds over C/I for every nonzero ideal I of C. Thus, by Remark 6.2, C is an integral domain. If C is finite, then it is a field, and we are done by Theorem 5.6. So we may assume that C is an infinite integral domain. We need to show that any T-ideal generated

by a given set of polynomials $\{g_1, g_2, \dots\}$ is finitely based. The coefficient ideals of $\{g_1, g_2, \dots, g_j\}$ stabilize to some ideal I of C at some j_0, since C is Noetherian. We let A_0 denote the relatively free algebra with respect to the T-ideal generated by $g_1, \dots, g_{j_0}$, Inductively, we let A_i denote the relatively free algebra with respect to the T-ideal generated by $f_{j_0+1}, \dots, f_{j_0+i}$, and take a PI-proper polynomial f_{i+1}, not in $\mathrm{id}(A_i)$ such that cf_{i+1} is in the T-ideal generated by g_{i+1} in A_i for all c in the coefficient ideal of g_{i+1}. (Such a polynomial exists in view of Proposition 7.4.) This gives us an ascending chain of PI-proper T-ideals of A_0, which must terminate in view of Theorem 6.22, a contradiction.   □

7.0.1. *Digression: Consequences of torsion for relatively free algebras.* Torsion has been so useful in this paper that we collect a few more elementary properties and apply them to relatively free algebras.

Lemma 7.7. *Suppose C is a Noetherian integral domain, and A is a relatively free affine C-algebra.*
   (1) *A has p-torsion for only finitely many prime numbers p.*
   (2) *There is some k_0 such that p^k-$\mathrm{tor}(A) = p^{k+1}$-$\mathrm{tor}(A)$ for all $k > k_0$ and all prime numbers p.*
   (3) *Let $\phi_k \colon A \to A \otimes \mathbb{Z}/p^k\mathbb{Z}$ denote the natural homomorphism. If $p^k A \neq p^{k+1}A$, then $\ker \phi_k \neq \ker \phi_{k+1}$.*

Proof. p^k-$\mathrm{tor}(A)$ is a T-ideal for each k. Let $\mathcal{I}_k$ be the T-ideal generated by p^k-torsion elements. The $\mathcal{I}_k$ stabilize for some k_0, yielding (2), and (3) follows since once the chain stabilizes we have $p^k A = p^{k+1}A$. Likewise, the direct sum of these T-ideals taken over all primes stabilizes, yielding (1).   □

7.1. **Applications to relatively free algebras.** As Kemer [19] noted, the ACC on T-ideals formally yields a Noetherian-type theory. We apply this method to Theorem 7.6.

Proposition 7.8. *Any relatively free algebra A over a commutative Noetherian ring has a unique maximal nilpotent T-ideal $N(A)$.*

Proof. By ACC, there is a maximal nilpotent T-ideal, which is unique since the sum of two nilpotent T-ideals is a nilpotent T-ideal.   □

Definition 7.9. The ideal $N(A)$ of Proposition 7.8 is called the **T-radical**. An algebra A is **T-prime** if the product of two nonzero T-ideals is nonzero. A T-ideal $\mathcal{I}$ of A is **T-prime** if $A/\mathcal{I}$ is a T-prime algebra.

Proposition 7.10. *The T-radical is the intersection of a finite number of T-prime T-ideals.*

Proof. Each T-prime T-ideal contains the T-radical, which we thus can mod out. Then just copy the usual argument using Noetherian induction.   □

   Kemer characterized all T-prime algebras of characteristic 0; cf. [5, Theorem 6.64]. The situation in nonzero characteristic is much more difficult, but in general we can reduce to the field case.

Proposition 7.11. *Each T-prime, relatively free algebra A with 1 over a commutative Noetherian ring C is either the free C-algebra or is PI-equivalent to a relatively free algebra over a field. In particular, either A is free or PI.*

Proof. The center Z of A is an integral domain over which A is torsionfree, since if $c \in C$ has torsion, then $0 = (cA) \operatorname{Ann}_A(c)$ implies $cA = 0$, so $c = 0$. If Z is finite, then it is a field and we are done. If Z is infinite, then A is PI-equivalent to $A \otimes ZK$ where K is the field of fractions of Z. □

We see that this theory, in particular Corollary 7.5, provides a method for generalizing results about relatively free PI-algebras to relatively free algebras in a variety which is not necessarily PI-proper. For example, let us generalize a celebrated theorem of Braun[11]-Kemer-Razmyslov:

Theorem 7.12. *The Jacobson radical J of any relatively free affine algebra A is nilpotent.*

Proof. Modding out the T-radical, and applying Proposition 7.10, we may assume that A is T-prime. If it is free, then $J = 0$, so we may assume that A is PI, where we are done by Braun's Theorem. □

Of course there is no hope to generalize this result to nonrelatively free algebras, since the nilradical of an affine algebra need not be nilpotent.

References

[1] S. A. Amitsur, *On the characteristic polynomial of a sum of matrices*, Linear and Multi-linear Algebra **8** (1979/80), no. 3, 177–182, DOI 10.1080/03081088008817315. MR560557 (82a:15014)

[2] A. Ya. Belov, *Counterexamples to the Specht problem* (Russian, with Russian summary), Mat. Sb. **191** (2000), no. 3, 13–24, DOI 10.1070/SM2000v191n03ABEH000460; English transl., Sb. Math. **191** (2000), no. 3-4, 329–340. MR1773251 (2001g:16043)

[3] A. Ya. Belov, *Local finite basis property and local representability of varieties of associative rings* (Russian, with Russian summary), Izv. Ross. Akad. Nauk Ser. Mat. **74** (2010), no. 1, 3–134, DOI 10.1070/IM2010v074n01ABEH002481; English transl., Izv. Math. **74** (2010), no. 1, 1–126. MR2655238 (2011e:16039)

[4] A. Ya. Belov, *On rings asymptotically close to associative rings [translation of MR2485366]*, Siberian Adv. Math. **17** (2007), no. 4, 227–267, DOI 10.3103/S1055134407040013. MR2643374

[5] Alexei Kanel-Belov and Louis Halle Rowen, *Computational aspects of polynomial identities*, Research Notes in Mathematics, vol. 9, A K Peters Ltd., Wellesley, MA, 2005. MR2124127 (2006b:16001)

[6] Alexei Belov-Kanel, Louis Rowen, and Uzi Vishne, *Structure of Zariski-closed algebras*, Trans. Amer. Math. Soc. **362** (2010), no. 9, 4695–4734, DOI 10.1090/S0002-9947-10-04993-7. MR2645047 (2012m:16035)

[7] Alexei Belov-Kanel, Louis H. Rowen, and Uzi Vishne, *Full quivers of representations of algebras*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5525–5569, DOI 10.1090/S0002-9947-2012-05565-6. MR2931338

[8] Alexei Belov-Kanel, Louis H. Rowen, and Uzi Vishne, *PI-varieties associated to full quivers of representations of algebras*, Trans. Amer. Math. Soc. **365** (2013), no. 5, 2681–2722, DOI 10.1090/S0002-9947-2012-05709-6. MR3020112

[9] A. Belov, L. H. Rowen, and U. Vishne, *Application of full quivers to polynomial identities*, Comm. in Alg., to appear (2012), 26 pp.

[10] George M. Bergman and Warren Dicks, *On universal derivations*, J. Algebra **36** (1975), no. 2, 193–211. MR0387353 (52 #8196)

[11] Amiram Braun, *The nilpotency of the radical in a finitely generated PI ring*, J. Algebra **89** (1984), no. 2, 375–396, DOI 10.1016/0021-8693(84)90224-2. MR751151 (85m:16007)

[12] Vesselin Drensky, *On the Hilbert series of relatively free algebras*, Comm. Algebra **12** (1984), no. 19-20, 2335–2347, DOI 10.1080/00927878408823112. MR755919 (86f:16005)

[13] A. V. Grishin, *Examples of T-spaces and T-ideals of characteristic 2 without the finite basis property* (Russian, with English and Russian summaries), Fundam. Prikl. Mat. **5** (1999), no. 1, 101–118. MR1799541 (2002a:16028)

[14] A. V. Il′tyakov, *Finiteness of the basis of identities of a finitely generated alternative PI-algebra over a field of characteristic zero* (Russian), Sibirsk. Mat. Zh. **32** (1991), no. 6, 61–76, 204, DOI 10.1007/BF00971199; English transl., Siberian Math. J. **32** (1991), no. 6, 948–961 (1992). MR1156745 (93c:17056)

[15] A. V. Iltyakov, *Polynomial identities of Finite Dimensional Lie Algebras*, monograph (2003).

[16] Nathan Jacobson, *Basic algebra. II*, W. H. Freeman and Co., San Francisco, Calif., 1980. MR571884 (81g:00001)

[17] A. R. Kemer, *Representability of reduced-free algebras* (Russian), Algebra i Logika **27** (1988), no. 3, 274–294, 375, DOI 10.1007/BF01978562; English transl., Algebra and Logic **27** (1988), no. 3, 167–184 (1989). MR997959 (90e:16027)

[18] A. R. Kemer, *Identities of finitely generated algebras over an infinite field* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 4, 726–753; English transl., Math. USSR-Izv. **37** (1991), no. 1, 69–96. MR1073084 (91j:16027)

[19] Alexander R. Kemer, *Identities of associative algebras*, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 351–359. MR1159223 (93d:16027)

[20] Alexander Kemer, *On some problems in PI-theory in characteristic p connected with dividing by p*, Proceedings of the Third International Algebra Conference (Tainan, 2002), Kluwer Acad. Publ., Dordrecht, 2003, pp. 53–66. MR2026093 (2004k:16062)

[21] Jacques Lewin, *A matrix representation for associative algebras. I, II*, Trans. Amer. Math. Soc. **188** (1974), 293–308; ibid. **188** (1974), 309–317. MR0338081 (49 #2848)

[22] Louis Halle Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics, vol. 84, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. MR576061 (82a:16021)

[23] Louis H. Rowen, *Ring theory. Vol. II*, Pure and Applied Mathematics, vol. 128, Academic Press Inc., Boston, MA, 1988. MR945718 (89h:16002)

[24] Louis Halle Rowen, *Graduate algebra: noncommutative view*, Graduate Studies in Mathematics, vol. 91, American Mathematical Society, Providence, RI, 2008. MR2462400 (2009k:16001)

[25] L. M. Samoĭlov, *Prime varieties of associative algebras and related nil-problems*, Dr. of Sci. dissertation, Moscow, 2010. 161 pp.

[26] V. V. Shchigolev, *Examples of infinitely basable T-spaces* (Russian, with Russian summary), Mat. Sb. **191** (2000), no. 3, 143–160, DOI 10.1070/SM2000v191n03ABEH000467; English transl., Sb. Math. **191** (2000), no. 3-4, 459–476. MR1773258 (2001f:16044)

[27] A. Ya. Vaĭs and E. I. Zel′manov, *Kemer's theorem for finitely generated Jordan algebras* (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. **6** (1989), 42–51; English transl., Soviet Math. (Iz. VUZ) **33** (1989), no. 6, 38–47. MR1017777 (90m:17042)

[28] M. R. Vaughan-Lee, *Varieties of Lie algebras*, Quart. J. Math. Oxford Ser. (2) **21** (1970), 297–308. MR0269710 (42 #4605)

[29] A. N. Zubkov, *Matrix invariants over an infinite field of finite characteristic* (Russian, with English and Russian summaries), Sibirsk. Mat. Zh. **34** (1993), no. 6, 68–74, ii, viii, DOI 10.1007/BF00973469; English transl., Siberian Math. J. **34** (1993), no. 6, 1059–1065. MR1268158 (95c:13003)

[30] K. A. Zubrilin, *Algebras that satisfy the Capelli identities* (Russian, with Russian summary), Mat. Sb. **186** (1995), no. 3, 53–64, DOI 10.1070/SM1995v186n03ABEH000021; English transl., Sb. Math. **186** (1995), no. 3, 359–370. MR1331808 (96c:16032)

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