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A.Ya. Belov

The following problem has been known since the last century: N hyperplanes in general position partition $\mathbb{R}^K$ into parts. Prove that there are $N-K$ simplices among the monolithic parts of the partition. It is not complicated to give examples of partitions with exactly this number of simplices. In 1979 Grünbaum and Shephard solved this problem, but their solution was not elementary and was quite long. In this paper we give a shorter elementary solution of this problem, where the general position condition is weakened.

Theorem. The space $\mathbb{R}^K$ $(K > 1)$ is partitioned by N hyperplanes, any K normals to which are linearly independent and there are no points common to all of them. Then there are $N-K$ simplices among the monolithic parts of the partition.

Proof. First we consider the case of general position. Let $\sum_j a_{ij}x_j = c_i$ be the equation of the i-th hyperplane. We may assume that $\sum_j a_{ij}^2 = 1$. We shall move the i-th hyperplane with velocity v_i, that is, we consider the sets of hyperplanes $\left\{\sum_j a_{ij}x_j = c_i + v_i t\right\}$ parametrized by the parameter t.

Suppose that the hyperplanes numbered $i_1, \ldots, i_K$ form a simplex. In the process of motion it generally changes its size, but it remains similar to the original one. The condition of constancy of size is a linear condition on the velocities $V_{i_1}, \ldots, V_{i_K}$:

$$\mathrm{DET}\begin{pmatrix} a_{1i_1} & \cdots & a_{Ki_1}V_{i_1} \\ & \cdots & \\ a_{1i_K} & \cdots & a_{Ki_K}V_{i_K} \end{pmatrix} = 0.$$

Now we assume that there are fewer than $N-K$ simplices. Fixing their sizes gives fewer than $N-K$ homogeneous equations on V_i. In order to avoid uninteresting parallel translations of the whole system we set $V_1 = V_2 = \ldots = V_K = 0$. As a result we have fewer than N homogeneous linear equations on the N parameters V_i. These conditions are not rigid, so the system can be moved. Changing all the signs of the velocities to their opposites if necessary, we may assume that some i-th hyperplane approximates the point of intersection of the first K. We consider the moment of first catastrophe t, when $K+1$ or more hyperplanes pass through a single point. At the moment $t-dt$, preceding the catastrophe, we have a family of more than K hyperplanes, passing close to some point but not having a common point. Among the parts of the partition of space that they form there will be at least one simplex (it suffices to consider an intersection point, close to the $(K+1)$-st hyperplane, of any K others). But as $dt \to 0$ the size of this simplex tends to 0, which is impossible, since the sizes of all the simplices were fixed.

The case of non-general position is considered analogously: we impose conditions on the simplices and the first K hyperplanes and again carry out an argument from the first catastrophe, by which we understand the appearance of a new focus. In the process of motion the old foci can be destroyed and new simplices can appear; however, it remains to observe that in the case of dimension >1 the new simplices in the process of motion may only increase their sizes.

The considerations used in the proof of Theorem 1 can be applied in a number of cases. We give several well-known examples together with sketches of the proofs.

Example 1. A rectangle is cut into squares. Prove that the ratio of its sides is rational.

Sketch of the proof. Assume the contrary. The partition of a rectangle into squares is described by a system of linear equations on the common pieces of the sides of the squares. Suppose that one side of the rectangle is equal to 1. The existence of solutions of the system of linear equations with an irrational parameter means the absence of rigidity, that is, the sides of the squares can vary in a compatible fashion (see [2]). We call the squares that do not come together at the moment of first catastrophe *big*. The fact that a narrow gap can be filled only by a large number of squares leads to the fact that, since they have a common piece of a side at the moment of catastrophe, the big squares have it also before the catastrophe. We now consider the corners of the big squares. There we can observe the pieced-together rectangle, partitioned into a smaller number of squares than the original, so that in the process of motion the ratio of its sides does not change. It remains to consider its centre as a reference system, and to observe that after a catastrophe there will be no overlaps, and we will again obtain a partition into squares. Thus all the catastrophes take place, which cannot happen.

Example 2. Let $0 \leqslant a_0 < \ldots < a_n = 1$. Then the trigonometric polynomial $P(x) = a_0 + a_1 \cos(x) + \ldots + a_n \cos(nx)$ vanishes exactly n times on the interval $[0, \pi]$.

Sketch of the proof. If there is an example that does not have this number of zeros, then in the process of transformation to a standard form a multiple root ξ arises. The conditions $P(\xi) = P'(\xi) = 0$ are linear conditions on the coefficients a_i. Further moving the coefficients a_i so that the inequalities on them are preserved and ξ remains a multiple root, we would find that the coefficients a_i would form four groups of equal, valid subseries, where the first group is zero and the last is one. For this case the absence of a multiple root is verified directly.

Example 3. The round-off of a number is the replacement of it by any of the integers that enclose it. Prove that it is possible to round off the numbers from an $N \times K$ table so that the sums over the rows and the columns are also rounded off.

Example 4. If a rectangle is partitioned into rectangles, each of which has an integer side, then the whole rectangle also has an integer side. Prove this.

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