ISSN 1064–5624, Doklady Mathematics, 2006, Vol. 73, No. 3, pp. 340–343. © Pleiades Publishing, Inc., 2006.
Original Russian Text © E.R. Avakov, A.V. Arutyunov, A.F. Izmailov, 2006, published in Doklady Akademii Nauk, 2006, Vol. 408, No. 2, pp. 151–154.

MATHEMATICS

Necessary Conditions for an Extremum in 2-Regular Problems

E. R. Avakov[a], A. V. Arutyunov[b], and A. F. Izmailov[c]

Presented by Academician A.B. Kurzhanskii October 28, 2005

Received November 16, 2005

DOI: 10.1134/S1064562406030070

Let X and Y be Banach spaces, $f: X \to \mathbf{R}$ be a smooth function, $F: X \to Y$ be a smooth mapping, and $Q \subset Y$ be a closed convex set. Consider the minimization problem

$$f(x) \to \min, \\ x \in D = F^{-1}(Q) = \{x \in X \mid F(x) \in Q\}. \tag{1}$$

Traditional studies of problem (1) are based on the Lagrange principle. There is extensive literature concerning first- and second-order necessary conditions for a local extremum in problem (1) (see, e.g., [1, Chapter 3]). However, in the majority of these studies (except for several ones, which are discussed below), the local solution $\bar{x}$ to problem (1) is assumed to satisfy the Robinson regularity condition

$$0 \in \operatorname{int}(F(\bar{x}) + \operatorname{im}F'(\bar{x}) - Q), \tag{2}$$

where int denotes the interior of a set and im stands for the image of a linear operator. In this case, we use the Lagrangian

$$L(x, \lambda) = f(x) + \langle \lambda, F(x) \rangle, \tag{3}$$

where $x \in X$ and $\lambda \in Y^*$. At the same time, in the irregular case, i.e., when (2) is violated, the necessary conditions for an extremum that use the function L from (3) are generally invalid.

When the relative interior of $F(\bar{x}) + \operatorname{im}F'(\bar{x}) - Q$ is nonempty (in particular, when Y is finite-dimensional), the irregular case can be formally covered by the Lagrange principle if we introduce an additional multiplier $\lambda_0 \in \mathbf{R}$ corresponding to the objective function (see [1, Proposition 3.18]). However, this generalization gives no additional information on the irregular situation, since the corresponding first-order necessary condition is then automatically satisfied with $\lambda_0 = 0$, irrespective of f (see [1, Proposition 3.16]).

The following generalized Lagrangian of problem (1) was introduced in [2, 3]:

$$L_2(x, h, \lambda^1, \lambda^2) = f(x) + \langle \lambda^1, F(x) \rangle \\ + \langle \lambda^2, F'(x)h \rangle, \tag{4}$$

where x, $h \in X$ and λ^1, $\lambda^2 \in Y^*$ play the role of Lagrange multipliers. By using this function, first- and second-order necessary conditions for a local extremum were obtained in the irregular case for a problem with equality constraints (i.e., with $Q = \{0\}$). Here, h is a parameter ranging over a set defined by the first and second derivatives of F at $\bar{x}$. These constructions are underlain by the concept of 2-regularity, and its relevant generalization also plays a central role in this paper. Similar ideas were used in [4, 5] for the case of inequality constraints (i.e., when Q is a cone with a nonempty interior). However, the case of both equality and inequality constraints under violating Robinson condition (2) has not been examined thus far. The goal of this paper is to fill this gap. Note that, although problem (1) is considered in a very general setting, the results presented below are meaningful for mathematical programming problems (which correspond to the case of polyhedral Q).

For any cone $K \subset X$, let $K^\circ = \{l \in X^* \mid \langle l, \xi \rangle \le 0 \ \forall \xi \in K\}$ denote its polar cone. For a given set $S \subset X$, let $S^\perp = \{l \in X^* \mid \langle l, x \rangle = 0 \ \forall X \in S\}$ denote its annihilator, $\sigma(l, S) = \sup_{x \in S} \langle l, x \rangle$ denote the support function of this set, and $\operatorname{dist}(x, S) = \inf_{\xi \in S} \|x - \xi\|$ is the distance from the point $x \in X$ to S. The radial cone for S at a point $x \in S$

[a] Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya ul. 65, Moscow, 117997 Russia
e-mail: era@maxmin.ru
[b] Russian University of Peoples' Friendship, ul. Miklukho-Maklaya 6, Moscow, 117198 Russia
e-mail: arutun@orc.ru
[c] Faculty of Computational Mathematics and Cybernetics, Moscow State University, Leninskie gory, Moscow, 119992 Russia
email: izmaf@ccas.ru

is defined by $R_S(x) = \text{cone}(S - x)$, the contingent cone is defined as

$$T_S(x) = \{h \in X | \exists \{t_k\} \subset \mathbf{R}_+ \backslash \{0\} \text{ such that } \{t_k\} \to 0, \quad \text{dist}(x + t_k h, S) = o(t_k)\},$$

and the normal cone is defined by the equality $N_S(x) = (T_S(x))^\circ$ (if $x \notin S$, then $N_S(x) = \varnothing$ by definition).

2-REGULARITY CONDITION AND THE CONTINGENT CONE

In what follows, we assume that the mapping F is twice Fréchet differentiable in the neighborhood of a given point $\bar{x} \in D$ and its second derivative is Lipschitz continuous in this neighborhood.

Definition 1. The mapping F is called 2-regular at $\bar{x}$ with respect to Q in a direction $h \in X$ if

$$0 \in \text{int}(F(\bar{x}) + \text{im}F'(\bar{x}) + F''(\bar{x})[[h,](F'(\bar{x}))^{-1}(Q - F(\bar{x}))] - Q). \qquad (5)$$

Note that 2-regularity in the direction $h = 0$ coincides with Robinson condition (2). Under the last condition, F is 2-regular at $\bar{x}$ with respect to Q in any direction $h \in X$ (including $h = 0$), but the converse is not valid. On the other hand, the concept of 2-regularity coincides with the corresponding concept introduced in [2] for the case of equality constraints (i.e., for $Q = \{0\}$).

The results stated below are based on the following estimate lemma.

Lemma 1. *Let F be 2-regular at $\bar{x}$ with respect to Q in a direction $h \in X$.*

Then there is $c = c(\bar{x}, h) > 0$ such that, for any $\tilde{h} \in X$ that is sufficiently close to h and for any sufficiently small $t > 0$, we have the estimate

$$\text{dist}(\bar{x} + t\tilde{h}, D) \le \frac{c\,\text{dist}(F(\bar{x} + t\tilde{h}), Q)}{t}.$$

For a given linear continuous operator A: $X \to Y$, a point $y \in Q$, and an element $d \in Y$, we define the set

$$T_Q^2(y, d; A) = \Big\{ w \in Y | \exists \{t_k\} \subset \mathbf{R}_+ \backslash \{0\}, \{x^k\} \subset X \text{ such that } \{t_k\} \to 0, \{x^k\} \to 0, \text{dist}\Big(y + t_k d + t_k A x^k + \frac{1}{2} t_k^2 w, Q\Big) = o(t_k^2) \Big\}.$$

In particular, $T_Q^2(y, d) = T_Q^2(y, d; 0)$ coincides with the usual (external) second-order tangent set of Q at $y \in Q$ in the direction $d \in Y$ as introduced in [1, Definition 3.28].

It should be stressed that, if $T_Q^2(y, d; A) \neq \varnothing$, then we necessarily have $d \in T_Q(y)$.

Define the sets

$$H_2(\bar{x}) = \{h \in X | F''(\bar{x})[h, h] \in T_Q^2(F(\bar{x}), F'(\bar{x})h; F'(\bar{x}))\},$$

$$\bar{H}_2(\bar{x}) = \{h \in H_2(\bar{x}) | \text{ holds } (5)\}.$$

It is easy to verify that these sets are both cones. The following result is derived from Lemma 1.

Theorem 1. *It holds that*

$$\bar{H}_2(\bar{x}) \subset T_D(\bar{x}) \subset H_2(\bar{x}).$$

Theorem 1 was obtained in [2] in the case of equality constraints (i.e., for $Q = \{0\}$) and in [4, 5] for inequality constraints, when Q is a cone with a nonempty interior. Finally, in the case where Q is a cone and $F(\bar{x}) = 0$, Theorem 1 was derived in [6].

If Robinson condition (2) is satisfied, Theorem 1 is reduced to the classical result on the tangent cone (see, e.g., [1, Corollary 2.91]), which is expressed by the equality

$$T_D(\bar{x}) = \{h \in X | F'(\bar{x})h \in T_Q(F(\bar{x}))\}.$$

At the same time, Theorem 1 also makes sense if the Robinson condition is violated.

FIRST-ORDER NECESSARY CONDITIONS

The necessary conditions for a local extremum given in this section are first-order conditions in the sense that they use only the first derivative of the objective function f. It is assumed that f is Fréchet differentiable at $\bar{x}$. Theorem 1 immediately implies the following direct first-order necessary condition: if $\bar{x}$ is a local solution to problem (1), then

$$\langle f'(\bar{x}), h \rangle \ge 0 \ \forall h \in \bar{H}_2(\bar{x}).$$

In the rest of this section, we present first-order necessary conditions that are a further development of the Lagrange principle to the irregular case. In particular, they imply the direct condition.

The reduced critical cone of problem (1) at the point $\bar{x}$ is defined as

$$C_2(\bar{x}) = \{h \in H_2(\bar{x}) | \langle f'(\bar{x}), h \rangle \le 0\}.$$

Moreover, we define the cone

$$\bar{C}_2(\bar{x}) = C_2(\bar{x}) \cap \bar{H}_2(\bar{x}) = \{h \in C_2(\bar{x}) | \text{ holds } (5)\}.$$

Theorem 2. *Let* $\bar{x}$ *be a local solution to problem* (1).

Then, for any $h \in \bar{C}_2(\bar{x})$, *there is a Lagrange multiplier* $\lambda^2 = \lambda^2(h) \in Y^*$ *such that*

$$-f'(\bar{x}) - (F''(\bar{x})[h])^*\lambda^2 \in ((F'(\bar{x}))^{-1}(R_Q(F(\bar{x}))))^\circ, \quad (F'(\bar{x}))^*\lambda^2 = 0, \quad \lambda^2 \in N_Q(F(\bar{x})). \tag{6}$$

Here, $F''(\bar{x})[h]$ *is the linear operator defined by the formula* $F''(\bar{x})[h]x = F''(\bar{x})[h, x]$, $x \in X$.

Let the generalized Lagrangian of problem (1) be defined by formula (4). It can be shown that, if the cone $R_Q(F(\bar{x}))$ is closed and there is a closed linear subspace M of Y that satisfies

$$\operatorname{im} F'(\bar{x}) \subset M \subset \operatorname{im} F'(\bar{x}) - R_Q(F(\bar{x})) \tag{7}$$

and such that $(R_Q(F(\bar{x})))^\circ + M^\perp$ is a weakly* closed cone, then (6) is equivalent to the existence of $\lambda^1 = \lambda^1(h) \in Y^*$ such that

$$\frac{\partial L_2}{\partial x}(\bar{x}, h, \lambda^1, \lambda^2) = 0, \quad (F'(\bar{x}))^*\lambda^2 = 0, \quad \lambda^1 \in N_Q(F(\bar{x})), \quad \lambda^2 \in N_Q(F(\bar{x})) \tag{8}$$

(note that M is not involved in these conditions).

A subspace M with the required properties exists in some important special cases. For example, if Y is finite-dimensional and Q is polyhedral (the case of a mathematical programming problem), then we can set $M = \operatorname{im} F'(\bar{x})$. If Robinson condition (2) holds at $\bar{x}$, then we can set $M = Y$. However, the existence of such a subspace cannot be guaranteed in the general case. Therefore, the above argument does not imply the existence of $\lambda^1 \in Y^*$ that satisfies (8). Moreover, it is not known at present whether the existence of λ^1 with the indicated properties can be guaranteed in the general case. At the same time, a somewhat weaker assertion holds true.

Theorem 3. *Let* $\bar{x}$ *be a local solution to problem* (1).

Then, for any $h \in \bar{C}_2(\bar{x})$, *there is a Lagrange multiplier* $\lambda^2 = \lambda^2(h) \in Y^*$ *such that, for any closed linear subspace* M *of* Y *satisfying* (7), *there exists a Lagrange multiplier* $\lambda^1 = \lambda^1(h; M) \in Y^*$ *such that*

$$\frac{\partial L_2}{\partial x}(\bar{x}, h, \lambda^1, \lambda^2) = 0,$$

$$(F'(\bar{x}))^*\lambda^2 = 0, \quad \lambda^1 \in N_{Q \cap (F(\bar{x}) + M)}(F(\bar{x})), \tag{9}$$

$$\lambda^2 \in N_Q(F(\bar{x})).$$

If Robinson condition (2) is fulfilled, then the second and fourth conditions in (9) imply that $\lambda^2 = 0$. Consequently, Theorem 3 becomes a traditional first-order necessary condition (see, e.g., [1, Theorem 3.9]). Specifically, if $\bar{x}$ is a local solution to problem (1), then there exists $\lambda \in Y^*$ such that

$$\frac{\partial L}{\partial x}(\bar{x}, \lambda) = 0, \quad \lambda \in N_Q(F(\bar{x})), \tag{10}$$

where the Lagrangian L is defined by (3). At the same time, the theorems stated in this section give meaningful information on $\bar{x}$ even if the Robinson condition is violated.

SECOND-ORDER NECESSARY CONDITIONS

This section deals with second-order necessary conditions (i.e., conditions that use the second derivative of f). It is assumed that f is twice Fréchet differentiable at $\bar{x}$ and F is three times Fréchet differentiable at this point.

For a given linear continuous operator $A: X \to Y$, a linear subspace $M \subseteq Y$, a point $y \in Q$, and elements d, $\eta \in Y$, we define the set

$$T_Q^3(y, d; A; M, \eta) = \Big\{(w^1, w^2) \in (\eta + M) \times Y \mid \exists \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \text{ such that } \{t_k\} \to 0, \quad \{x^k\} \to 0,$$
$$\operatorname{dist}\Big(y + t_k d + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 A x^k + \frac{1}{3!}t_k^3 w^2, Q\Big) = o(t_k^3)\Big\},$$

and the set

$$T_Q^3(y, d; A) = T_Q^3(y, d; A; Y, \eta)$$

(which is independent of the choice of η).

For every $h \in X$, define the set

$$\Xi(\bar{x}, h) = \{\xi \in X \mid \exists x \in X \text{ such that } (F'(\bar{x})\xi + F''(\bar{x})[h, h],\ F'(\bar{x})x + 3F''(x)[h, \xi] + F'''(x)[h, h, h]) \in T_Q^3(F(\bar{x}), F'(\bar{x})h; F'(\bar{x}))\}.$$

The following direct second-order necessary condition is valid: if $\bar{x}$ is a local solution to problem (1), then, for any $h \in \bar{C}_2(\bar{x})$,

$$\langle f'(\bar{x}), \xi \rangle + f''(\bar{x})[h, h] \geq 0 \ \forall \xi \in \Xi(\bar{x}, h).$$

If Robinson condition (2) is satisfied, this result is reduced to a well-known one (see, e.g., [1, Lemma 3.44]). Now we proceed to the Lagrangian form of the second-order necessary condition.

Theorem 4. *Let $\bar{x}$ be a local solution to problem* (1).

Then, for any closed linear subspace M of Y satisfying (7), *for any $h \in \bar{C}_2(\bar{x})$, and any convex set*

$$\mathcal{T} \subset T_Q^3(F(\bar{x}), F'(\bar{x})h;\ F'(\bar{x});\ M, F''(\bar{x})[h, h]), \quad (11)$$

there exist Lagrange multipliers $\lambda^1 = \lambda^1(h; M) \in Y^*$ and $\lambda^2 = \lambda^2(h; M) \in Y^*$ such that (9) *holds and*

$$\frac{\partial^2 L_2}{\partial x^2}\left(\bar{x}, h, \lambda^1, \frac{1}{3}\lambda^2\right)[h, h] - \sigma((\lambda^1, \lambda^2), \mathcal{T}) \geq 0. \quad (12)$$

Proposition 1. *Let $\bar{x}$ be a local solution to problem* (1). *Assume that the cone $R_Q(F(\bar{x}))$ is closed and there is a closed linear subspace M of Y that satisfies* (7) *and is such that the cone $(R_Q(F(\bar{x})))^\circ + M^\perp$ is weakly* closed.*

Then Theorem 4 *holds with this M and condition* (9) *can be replaced by* (8).

In Robinson condition (2) is fulfilled, we can use $M = Y$ in Theorem 4 and the latter then becomes a traditional second-order necessary condition (see, e.g., [1, Theorem 3.45]). Specifically, if $\bar{x}$ is a local solution to problem (1), then, for any $h \in \bar{C}(\bar{x})$ and any convex set $T \subset T_Q^2(F(\bar{x}), F'(\bar{x})h)$, there is $\lambda = \lambda^1 = \lambda^1(h) \in Y^*$ such that (10) holds and

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \lambda)[h, h] - \sigma(\lambda, T) \geq 0. \quad (13)$$

Here,

$$C(\bar{x}) = \{h \in (F'(\bar{x}))^{-1}(T_Q(F(\bar{x})))\,|\ \langle f'(\bar{x}), h\rangle \leq 0\}$$

is the critical cone of problem (1) at $\bar{x}$.

It is well known that the so-called σ-term in (13) is always nonpositive (see [1, (3.109)]). The same holds true for the σ-term in (12), at least for those $\lambda^1 \in Y^*$ and $\lambda^2 \in Y^*$ for which (8) is satisfied (see Proposition 1).

Proposition 2. *For any closed linear subspace M of Y satisfying* (7), *any $h \in \bar{C}_2(\bar{x})$, any convex set $\mathcal{T} \subset T \times Y$ satisfying* (11), *and any $\lambda^1 \in Y^*$ and $\lambda^2 \in Y^*$ satisfying* (8), *we have the inequality*

$$\sigma((\lambda^1, \lambda^2), \mathcal{T}) \leq 0.$$

However, it should be noted that the σ-term in (12) is responsible only for the "curvature" of Q near $F(\bar{x})$. Accordingly, this term can be dropped if Q is polyhedral. Indeed, it is easy to verify that, in the latter case, the set $T_Q^3(F(\bar{x}), F'(\bar{x})h;\ M, F''(\bar{x})[h, h])$ contains the point (0, 0) and Theorem 4 can be applied when $\mathcal{T} = \{(0, 0)\}$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project nos. 05-01-00275, 05-01-00193, and 04-01-00341), the program "Leading Scientific Schools" (project nos. NSh-1889.2003.1 and NSh-1815.2003.1), and a grant from the President of the Russian Federation for supporting young Russian doctors of sciences (project no. MD-2723.2005.1).

REFERENCES

1. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems* (Springer-Verlag, New York, 2000).
2. E. R. Avakov, Zh. Vychisl. Mat. Mat. Fiz. **25**, 680–693 (1985).
3. E. R. Avakov, Mat. Zametki **45** (6), 3–11 (1989).
4. A. F. Izmailov, Zh. Vychisl. Mat. Mat. Fiz. **34**, 837–854 (1994).
5. A. F. Izmailov, Mat. Zametki **66** (1), 89–102 (1999).
6. A. V. Arutyunov, Mat. Zametki **77**, 483–497 (2005).